Ch. 1.4
Nested Quantifiers

Main idea: Quantifiers may appear within the scope of other quantifiers.

Example 1: Assume that the universe of discourse for the variables $x$ and $y$ consists of all real numbers. The statement $\forall x \forall y (x + y = y + x)$ says that for every real number $x$ and for every real number $y$, $x + y = y + x$. This is the ______________ law for addition of real numbers.

Example 2: Assume that the universe of discourse for the variables $x$ and $y$ consists of all real numbers. The statement $\forall x \exists y (x + y = 0)$ says that for every real number $x$ there exists a real number $y$, such that $x + y = 0$. This is the ______________ law for addition of real numbers.

Example 3: Assume that the universe of discourse for the variables $x$ and $y$ consists of all real numbers.
   i. Translate the statement $\exists x \forall y (x + y = 0)$ to English.

   ii. What is the truth value of $\exists x \forall y (x + y = 0)$.

The order of quantifiers of is important unless all the quantifiers are universal of all existential. As we have seen in the previous two examples $\exists x \forall y P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent.

$\forall x \exists y P(x, y)$

$\exists x \forall y P(x, y)$
**Example 4:** Translate the statement “The sum of two positive integers is positive” into a logical expression.

**Example 5:** Translate the statement “Every real number except zero has a multiplicative inverse” into a logical expression.

**Example 6:** Assume the universe of discourse is the set of elements a set $U$. Let $A$ and $B$ be subsets of $U$. A function from $A$ to $B$ is one-to-one if for every $a$ and $b$ in $A$, $a=b$ whenever $f(a) = f(b)$.

i) Express the definition in terms of logical connectives and quantifiers.

ii) Using part i) explain when a function is not one-to-one.
Example 7: Determine the truth value of each of the following statements if the universe of discourse for all variables consists of all integers.

a) \( \forall n \exists m(n^2 < m) \)

b) \( \exists n \forall m(n^2 < m) \)

c) \( \neg \exists n \forall m(n^2 < m) \)

d) \( \exists n \forall m(nm = m) \)

e) \( \exists n \forall m(nm = m) \)

f) \( \exists n \exists m(n^2 + m^2 = 5) \)

g) \( \exists n \exists m(n^2 + m^2 = 6) \)

f) \( \exists n \exists m(n + m = 4 \land n - m = 1) \)

h) \( \exists n \exists m(n + m = 4 \land n - m = 2) \)

i) \( \forall n \forall m \neg (n + m = 4 \land n - m = 2) \)

j) \( \forall n \forall m(n + m \neq 4 \lor n - m \neq 2) \)