Logic

Def. A **Proposition** is a statement that is either true or false.

**Examples:** Which of the following are propositions?

<table>
<thead>
<tr>
<th>Statement</th>
<th>Proposition (yes or no)</th>
<th>If yes, then determine if it is true or false.</th>
</tr>
</thead>
<tbody>
<tr>
<td>UHD has a brick exterior</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 + 3 = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 + 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x + 7 = 18$ for $x = 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x + 7 = 18$ for every real number $x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notation:** We use lower case letters to denote propositions.

Def. **Compound propositions** are new propositions formed from existing propositions using *logical operators*.

Note: We will discuss the following *logical operators* (also called connectives)
- Negation operator, $\neg$ (Other books use $\sim$)
- Conjunction operator, $\wedge$
- Disjunction operator, $\vee$
- Exclusive or, $\oplus$
- Implication, $\rightarrow$
- Biconditional, $\leftrightarrow$

Def. Let $p$ be a proposition. The statement "it is not the case that $p$" is a proposition formed from $p$ and the negation operator, called the **negation of** $p$, which we denote by $\neg p$. This proposition is read as "not $p$". The truth-value of $\neg p$ is true when $p$ is false and false when $p$ is true.

**Truth Table**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Examples:
1. Let $p$ be the statement "$7 + 1 = 6$".
   a. Write the statement $\neg p$ as an English statement.

   b. What is the truth-value of $\neg p$?

2. Let $p$ be the statement "It is not Friday".
   a. Write the statement $\neg p$ as an English statement.

   b. What is the truth-value of $\neg p$?

Def. Let $p$ and $q$ be propositions. The statement "$p$ and $q$" is a proposition formed from $p$, $q$ and the conjunction operator, called the conjunction of $p$ and $q$, which we denote by $p \land q$. The truth-value of $p \land q$ is true when both $p$ and $q$ are true, otherwise $p \land q$ is false.

\[
\begin{array}{ccc}
p & q & p \land q \\
T & T & T \\
T & F & F \\
F & T & F \\
F & F & F \\
\end{array}
\]

Def. Let $p$ and $q$ be propositions. The statement "$p$ or $q$" is a proposition formed from $p$, $q$ and the disjunction operator, called the disjunction of $p$ and $q$, which we denote by $p \lor q$. The truth-value of $p \lor q$ is false when both $p$ and $q$ are false, otherwise $p \lor q$ is true.

\[
\begin{array}{ccc}
p & q & p \lor q \\
T & T & T \\
T & F & T \\
F & T & T \\
F & F & F \\
\end{array}
\]

Examples: Let $p$ be the statement "Houston is the capital of Texas" and let $q$ be the statement "Houston is a city in Texas". Determine the following:

a. The truth-value for $p \land q$.

b. The truth-value for $p \lor q$.

c. The truth value for $\neg p \land q$

d. The truth value for $\neg (p \land q)$

e. The truth value for $\neg p \lor q$
Def. Let \( p \) and \( q \) be propositions. The statement "\( p \) or \( q \) exclusively" is a proposition formed from \( p, q \) and the exclusive or operator, called the **exclusive or of \( p \) and \( q \)**, which we denote by \( p \oplus q \). The truth-value of \( p \oplus q \) is true when exactly one of \( p \) or \( q \) is true, otherwise \( p \oplus q \) is false.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( p \oplus q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

**Truth Table**

Comparison: A familiar example of the use of an "exclusive or" is a restaurant menu. "Price of the entree includes soup or salad". Note that one or the other is included in the price **but not both**. A familiar example of the use of inclusive or is the prerequisite for this course. "Students who have taken calculus or computer science, can take this course." Note that if a student has taken both courses, then the student may still take this course.

Def. An **implication** (or **conditional statement**) is statement of the form "if \( p \) then \( q \)"\), denoted \( p \rightarrow q \). Such propositions are also read as "\( p \) implies \( q \)". The "if part" is called the **hypothesis** (or **premise**), and the "then part" is called the **conclusion** (or **conclusion**). The implication if \( p \) then \( q \) is false only when the hypothesis is true and the conclusion is false.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
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<td>F</td>
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</tbody>
</table>

**Terminology**: An implication \( p \rightarrow q \) may be worded in several different ways:

- If \( p \), then \( q \)
- If \( p, q \)
- \( q \) if \( p \)
- \( p \) only if \( q \)
- \( p \) is sufficient for \( q \)
- \( q \) is necessary for \( p \)
Examples:
  a. True or False: If 2 is an even integer, then today is Friday.

  b. Write the statement in the form “if $p$, then $q$” and determine the truth-value. $1 + 1 = 2$ if $3 + 3 = 7$.

  c. Write the statement in the form “if $p$, then $q$” and determine the truth-value. $1 + 1 = 2$ only if $3 + 3 = 7$.

Def. Given the implication $p \rightarrow q$:

$q \rightarrow p$ is the converse of the implication $p \rightarrow q$.
$\neg p \rightarrow \neg q$ is the inverse of implication $p \rightarrow q$.
$\neg q \rightarrow \neg p$ is the contrapositive of implication $p \rightarrow q$.

Example: Write the converse, the inverse, and the contrapositive of the given implication:

<table>
<thead>
<tr>
<th>If $cows$ eat grass, then $2 + 3 = 4$.</th>
<th>Proposition</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converse</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inverse</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Contrapositive</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Def. The biconditional proposition denoted by $p \leftrightarrow q$ is the conjunction of $p \rightarrow q$ and $q \rightarrow p$. The proposition $p \leftrightarrow q$ is read as $p$ if and only if $q$. The biconditional proposition is true when both $p$ and $q$ have the same truth-values and is false otherwise.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Examples: Determine the truth-values for the following:

  a. The number 2 is an odd integer if and only if today is Friday. _____
  b. If $3 + 3 = 7$, $1 + 1 = 2$ and conversely if $1 + 1 = 2$, $3 + 3 = 7$. _____
Exercise. Let $p$ and $q$ be the propositions

$p$: It is below freezing.
$q$: It is snowing

Write the following propositions using the symbols $p$ and $q$ and any appropriate logical connectives.

Use the following (a) –(d) to fill in the blank for i) and ii)

(a) $p \land q$  (b) $p \lor q$  (c) $p \rightarrow q$  (d) none of these

i) It is below freezing and snowing. _______
ii) It is below freezing if it is snowing _______

Use the following (a) –(d) to fill in the blank for iii) - vi)

(a) $p \oplus q$  (b) $p \lor q$  (c) $p \rightarrow q$  (d) none of these

iii) It is not below freezing and it is not snowing. _______
iv) It is either snowing or below freezing (or both). _______
v) It is either snowing or below freezing (but not both). _______
vi) That it is below freezing is necessary and sufficient for it to be snowing. _______

Exercise  Construct a truth table for each of the following compound propositions. Note: your truth-table will require $2^n$ rows, where $n$ is the number of simple propositions in the compound proposition.

a) $(p \land q) \lor \neg q$

b) $(p \lor q) \land \neg r$