Prop #29

\((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \equiv T\)

Proof:

\((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \equiv \)

\(\neg ((p \rightarrow q) \land (q \rightarrow r)) \lor (p \rightarrow r) \equiv \)

\(\neg (\neg p \lor q) \lor (\neg q \lor r) \lor (p \rightarrow r) \equiv \)

\(\neg (\neg p \lor q) \lor (\neg q \lor r) \lor (\neg p \lor \neg q \lor r) \lor (p \rightarrow r) \equiv \)

\(\neg (\neg p \lor q) \lor (\neg q \lor r) \lor (\neg p \lor \neg q \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by De Morgan's

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (p \rightarrow r) \equiv \)

by De Morgan's X2

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Double Negation X2

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Distributivity

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Commutativity

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Negation

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Identity

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Association

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Commutativity

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Associativity

\((\neg p \lor q) \land (\neg q \lor r) \land (\neg p \lor r) \lor (\neg r \lor r) \lor (p \rightarrow r) \equiv \)

by Identity
\[
\vcenter{\hbox{\begin{align*}
\left( \overline{T \lor \overline{p \land q}} \right) \land \overline{r} & \equiv \text{Assoc.} \\
\left( \overline{T \lor \overline{p \land q}} \right) \land \overline{r} & \equiv \text{Assoc.} \\
\overline{T \lor \overline{p \land q}} \land \overline{r} & \equiv \text{Comm., Neg.} \\
\left( \overline{T \lor \overline{p \land q}} \right) \lor \overline{r} & \equiv \text{Dom.} \\
\overline{T \lor \overline{p \land q}} & \equiv T
\end{align*}}}\]
\]
Ch 1.4 Predicates and Quantifiers

Recall that the expression $x > 3$ is not a proposition. Why?

**$P(x)$ Notation:**
- We will use the propositional function notation $P(x)$ to denote the expression "$x$ has property $P$".
- As soon as $x$ is assume to be some value then $P(x)$ has a truth value.

The goal of the section is to quantify such statements so that the result is a proposition. The process is very much in the spirit of the next example.

**Example:** Let $P(x)$ denote the statement "$x > 3$".
What are the truth values:
- i. $P(4)$ True or False?
- ii. $P(2)$ True or False?
- iii. $P(y)$ True or False?

**Example:** Let $Q(x,y)$ denote the statement "$x = y + 3$".
What are the truth values:
- i. $Q(7,4)$ True or False?
- ii. $Q(2,2)$ True or False?

**Quantifiers**

**Def.** The **universe of discourse** for a math statement is the domain of that statement.

**Def.** The **universal quantification of $P(x)$** is the proposition "$P(x)$ is true for all values of $x$ in the universe of discourse."
- This proposition is denoted by $\forall x P(x)$.
- The proposition $\forall x P(x)$ is read as "for all $x$ $P(x)$" or "for every $x$ $P(x)$".
- The symbol $\forall$ is called the **universal quantifier**.

**Def.** The **existential quantification of $P(x)$** is the proposition "There exists an element $x$ in the universe of discourse for which $P(x)$ is true."
- This proposition is denoted by $\exists x P(x)$.
- The symbol $\exists$ is called the **existential quantifier**.
- The proposition $\exists x P(x)$ is read as "for some $x$ $P(x)$" or "there exists an $x$ such that $P(x)$".
The Truth Values of $\forall x P(x)$ and $\exists x P(x)$

Example 1: Suppose that the universe of discourse for $x$ is the set of nonnegative integers $\{0, 1, 2, 3, 4, \ldots\}$.

What is the truth value of $\forall x \ (x > 0)$? \(\underline{\text{false}}\) \(\underline{6/6 \ 0 \ \neq 0}\)

What is the truth value of $\exists x \ (x > 0)$? \(\underline{\text{true}}\) \(\underline{1 > 0}\)

Example 2: Suppose that the universe of discourse for $P(x)$ is the small set $\{ x_1, x_2, x_3, x_4 \}$.

Since $\forall x P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land P(x_4)$

when is $\forall x P(x)$ true? \(\text{when each } P(x_i) \text{ is true}\)

when is $\forall x P(x)$ false? \(\text{when at least one of the } P(x_i) \text{ is false}\)

Since $\exists x P(x) \equiv P(x_1) \lor P(x_2) \lor P(x_3) \lor P(x_4)$

when is $\exists x P(x)$ true? \(\text{when at least one of } P(x_i) \text{ is true}\)

when is $\exists x P(x)$ false? \(\text{when all } P(x_i) \text{ are false}\)

Note: The above observations also hold for infinite universe of discourse sets.

Examples 3:

<table>
<thead>
<tr>
<th>Where $P(x)$ denotes:</th>
<th>Universe of Discourse for $P(x)$</th>
<th>Truth value of $\exists x P(x)$</th>
<th>Truth value of $\forall x P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = x + 3$</td>
<td>Real numbers</td>
<td>false</td>
<td>false</td>
</tr>
<tr>
<td>$x = x + 0$</td>
<td>Real numbers</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>$0/x = 0$</td>
<td>Integers</td>
<td>true</td>
<td>false</td>
</tr>
<tr>
<td>$0 - x = x$</td>
<td>Positive Integers</td>
<td>true</td>
<td>false</td>
</tr>
<tr>
<td>$0 - x = x$</td>
<td>Natural numbers</td>
<td>true</td>
<td>false</td>
</tr>
</tbody>
</table>

Example 4:

(i) What is the truth value of $[\forall x P(x) \rightarrow \exists x P(x)]$? \(\text{true}\)

(ii) What is the truth value of $[\exists x P(x) \rightarrow \forall x P(x)]$? \(\text{false}\)
Example 5: Let \( P(x) \) denote the statement "\( x^2 = 4 \)". Suppose that the universe of discourse is the set of real numbers.

(a) For every real number \( y \) there is a real number \( x \) such that \( x^2 = 4 \).

(b) For every real number \( x \) and for every real number \( y \) it is the case that \( x^2 = 4 \).

(c) There is a real number \( y \) such that for every real number \( x \) it is the case that \( x^2 = 4 \).

Which of the above is the English equivalent to \( \exists x P(x) \)?

What is the truth value of \( \exists x P(x) \)?

Which of the above is the English equivalent to \( \forall x P(x) \)?

What is the truth value of \( \forall x P(x) \)? and why?

Definition. A real number \( x \) is a rational number if it can be expressed as the ratio of two integers \( a \) and \( b \) with \( b \) not zero, that is if \( x = \frac{a}{b} \), with \( b \neq 0 \). A real number that is not rational is called an irrational number.

Example 7: Give four examples of rational numbers \( \frac{1}{2}, \frac{3}{4}, \frac{5}{7}, \frac{7}{8} \)

Give four examples of irrational numbers \( \sqrt{2}, \sqrt{3}, \pi, \sqrt{5}, \sqrt{7}, \sqrt{11} \)

Example 8: Let \( Q(x) \) denote the statement "\( x \) is a rational number ", and \( I(y) \) denote the statement "\( y \) is an integer. Assume the universe of discourse is the set of real numbers.

i) Which of the following is the English translation of \( \exists y (Q(y) \land I(y)) \)?

(a) There is a rational number \( y \) which is a rational number and also an integer.

(b) There is a real number \( y \) which is a rational number and also an integer.

(c) Every real number \( y \) is a rational and also an integer.

ii) What is the truth value of \( \exists y (Q(y) \land I(y)) \) and why?
Example 9: Let \( Q(x) \) denote the statement " \( x \) is a rational number ", and \( I(y) \) denote the statement " \( y \) is an integer. Assume the universe of discourse is the set of real numbers.

i) Which of the following is the English translation of \( \forall y \ (Q(y) \land I(y)) \)?
(a) There is a rational number \( y \) which is a rational number and also an integer.
(b) There is a real number \( y \) which is a rational number and also an integer.
(c) Every real number \( y \) is a rational or it is an integer.
(d) Every real number \( y \) is a rational and also an integer.

ii) What is the truth value of \( \forall y \ (Q(y) \land I(y)) \) and why? \( \text{false} \)

iii) Which of the following is the English translation of \( \forall y \ (I(y) \rightarrow Q(y)) \)?
(a) Every real number \( y \) is an integer and also a rational number.
(b) There is a real number \( y \) such that if \( y \) is an integer then \( y \) is a rational number.
(c) For any real number \( y \), if \( y \) is an integer then \( y \) is a rational number.

iv) What is the truth value \( \forall y \ (I(y) \rightarrow Q(y)) \)? And why? \( \text{true} \)

v) What is the English translation of \( \forall y \ (\neg I(y) \rightarrow Q(y)) \)?

vi) What is logic notation of “If \( y \) is a non-integer, then it is rational.”? What is its truth value? And why?
Example 10: Assume the universe of discourse is the set of all students at UHD. Let \( C(x) \) be "\( x \) has a computer" and let \( F(x,y) \) be "\( x \) and \( y \) are friends."

i. Translate \( C(Judy) \) into English. __________

ii. Which is the English translation of \( \exists y (C(y) \land F(Judy,y)) \).

(a) Judy has a friend who has a computer.
(b) Judy has a computer and a friend.
(c) There is a student who has a friend and a computer.

iii. Translate \( C(Judy) \land \exists y (C(y) \land F(Judy,y)) \) into English.

iv. Translate \( \forall x (C(x) \lor \exists y (C(y) \land F(x,y)) \) into English.

Negations of Quantified Expressions

Example 10: Consider the proposition "Every student in the class has taken a course in calculus."

- When is this proposition true? _________________

- Write the proposition in symbolic logic notation.

- Which of the following is the negation of "Every student in the class has taken a course in calculus."
  (a) No student in the class has taken calculus.
  (b) Some students in the class have not taken calculus.
  (c) Some student in the class has not taken calculus

- Write the negation of the proposition in symbolic logic notation.

- When is the negation of the proposition "Every student in the class has taken a course in calculus" true? _________________
Review

Quantifiers

- \( P(x) \) is a proposition function
- \( \forall x \ P(x) \) "for all" universal quantifier
- \( \exists x \ P(x) \) "there exists" existential quantifier

**Skills!**
- Evaluate quantified statements
  - \( \forall x \ x + 1 = x \) where \( \mathbb{N} \) is \( \mathbb{Z} \)
- Translating symbols to English
- Translating English to symbols

When we left off, negating the quantified statements
Example 11:
(iii) What is the truth value of \((\forall x)(P(x))\)? ______________
(iv) What is the truth value of \((\forall x)(P(x))\)? ______________
(v) What is the truth value of \((\exists x)(P(x))\)? ______________

Example 12: Write a statement equivalent to \(\forall x\forall y (P(x,y) \lor Q(x,y))\) so that the negation appears only within predicates (i.e., not in front of quantifiers).