Ch. 4.4 Limits at Infinity

Recall from 2.2 Definition: We write
\[ \lim_{x \to a} f(x) = L \]
and say “the limit of \( f(x) \), as \( x \) approaches \( a \) is equal to \( L \)”

In this section we are interested in when it is the case that \( \lim_{x \to \infty} f(x) = L \)

**Definition:** Let \( f \) be a function defined on some interval \((a, \infty)\). Then
\[ \lim_{x \to \infty} f(x) = L \]
means that the values of \( f(x) \), can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large”

**Example:** Let us consider \( \lim_{x \to \infty} \frac{1}{x} \).

What happens to this function as \( x \) gets increases?

**Example:** Let us consider \( \lim_{x \to \infty} \frac{1}{x^2} \).

What happens to this function as \( x \) gets increases?
What happens to this function as \( x \) gets decreases?
**Theorem** If \( r > 0 \) is a rational number (ratio of integers), then \( \lim_{x \to \infty} \frac{1}{x^r} = 0 \).

If \( r > 0 \) is a rational number (ratio of integers) such that \( xr \) is defined for all \( x \), then \( \lim_{x \to \infty} \frac{1}{x^r} = 0 \).

**Strategy on using the Theorem for other limits**

**Example**: Evaluate \( \lim_{x \to \infty} \frac{3x^2}{x^2 - x} \).

**Example**: Evaluate \( \lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \).

**What these limits mean to the graph**

**Definition** The line \( y = L \) is called a horizontal asymptote of the curve \( y = f(x) \) if either \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \).
Example: This is the graph of \( f(x) = \frac{1}{x-3} \).

Does it have a horizontal asymptote? If so, what is it and use calculus to argue that it is a horizontal asymptote.

Recall from Ch. 2.2

Definition: The line \( x = a \) is called a vertical asymptote of the curve \( y = f(x) \) if at least one of the following statements is true:

\[
\lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty
\]

\[
\lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty
\]

Example: Find the horizontal and vertical asymptotes of the graph of the function \( f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5} \)
**Example:** Evaluate \( \lim_{x \to \infty} \sin x \)

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**Infinite Limits at Infinity**

What should \( \lim_{x \to \infty} f(x) = \infty \) mean?

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**Example:** Evaluate \( \lim_{x \to \infty} x^2 \) and \( \lim_{x \to -\infty} x^2 \)

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**Example:** Evaluate \( \lim_{x \to \infty} (x^2 - x) \)

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**Example:** Evaluate \( \lim_{x \to \infty} \frac{x^2 + x}{3 - x} \)