Introductory Graph Theory

I. Basic Concepts

1. Definition of a Graph

**Intuitive Definition:** A simple graph is a collection of vertices (visualized as dots) and edges (visualized as arcs between dots).

![Figure 1.1](image)

**Formal Definition:** A simple graph $G$ with $n$ vertices and $m$ edges consists of a vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$, where $E(G) \subseteq V(G) \times V(G)$, that is each edge is an unordered pair of vertices.

**Definitions:** A vertex $v$ is incident to an edge if $v$ is one of the pair of vertices which determines the edge. The degree of a vertex is the number of edges to which it is incident. We denote the degree of a vertex $v$ as $\deg(v)$. Given a graph $G$ on vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, the degree sequence of $G$ is $\deg(v_1), \deg(v_2), ..., \deg(v_n)$. The maximum value of the degree sequence is the maximum degree of the graph and we denote it by $\Delta(G)$. The minimum value of the degree sequence is the minimum degree of the graph and we denote it by $\delta(G)$.

**Example:** Suppose $G$ is a graph on vertex set $\{a, b, c, d, e\}$ and edge set $\{(a, b), (b, c), (c, d), (d, e), (d, a)\}$. Note that the maximum degree of the graph is 3 and the minimum degree is 1. This graph can be visualized using figure 1.1 by labeling the vertices and edges as follows:

![Figure 1.2](image)
2. Applications of Graphs

Example. Acquaintance Graph. Suppose that the people are vertices and that there is an edge between two people if they are acquaintances.

Determine each of the following for the graph above.
The number of vertices _____,
The maximum degree _____,
The minimum degree _____,

For other examples of graph applications see text. Web as a graph

3. Some Special Graphs and More Graph Concepts and Invariants

A graph concept is a property of the graph. A graph invariant is a numeric value associated with the graph, usually independent of the way the graph is drawn.

Definitions: Let $G$ be a graph on vertex set $V$ and edge set $E$. For any two vertices, say $u$ and $v$, if $(u, v)$ is an edge we say that $u$ is adjacent to $v$. To indicate that two vertices $u$ and $v$ are adjacent we use the notation $u \sim v$. Further if $e = (u, v)$ is an edge, we say that edge $e$ is incident with vertex $u$, and also that the vertex $u$ is incident with edge $e$.

The empty graph on $n$ vertices (also called the null graph on $n$ vertices) is the graph on $n$ vertices with no edges.
The complete graph on \( n \) vertices is the graph on \( n \) vertices in which every two vertices are adjacent. We use the notation \( K_n \) to denote the complete graph on \( n \) vertices.

Perhaps you wondered why we introduced no notation for the empty graph? The following definition provides one motivation.

**Definition:** Let \( G \) be a graph on vertex set \( V \) and edge set \( E \). We define the complement graph of \( G \), denoted \( \overline{G} \), as a graph on the same vertex set \( V \) in which two vertices adjacent in \( G \) if and only if they are not adjacent in \( G \).

With some thought it is easily seen that the empty graph on \( n \) vertices is denoted by \( K_n^\circ \).

**Exercise:** Draw \( \overline{G} \) if \( G \) is the graph drawn below.

**Exercise:** Find a graph \( G \) (on four vertices) for which \( G \) and \( \overline{G} \) can be drawn so they appear the same; formally this should be so that \( G \cong \overline{G} \). The symbol \( \cong \) is used to indicate that the two graphs are essentially the same except for the way the vertices are labeled (formally the symbol \( \cong \) is read as isomorphic to, which we will investigate soon.)

**Exercise:** Suppose we are given a graph on vertex set \( \{0, 1, 2, 3, 4\} \) and that edges of this graph are determined by the following rule:

\[
\forall x, y \in \{0, 1, 2, 3, 4\} \quad x \sim y \iff x - y = 1 \text{ or } x - y = -1
\]

Draw the edges subject to the given rule in the following diagram.

```
0   1   2   3   4
●   ●   ●   ●   ●
```

The graph in the above Exercise is an example (with \( n = 5 \)) of a graph
called a \textit{path on }n\textit{ vertices}, denoted }P_n\textit{. Your diagram should suggest how such graphs inherited their names. In the example you were exposed to a rule that resulted a Path on 5 vertices. However, a definition of such graphs is usually given as follows.

\textbf{Definition:} A graph is called a \textit{Path on }n\textit{ vertices} if the vertices can be labeled with elements of \(\{0, 1, 2, \ldots, n-1\}\) so that the edge set is 
\[\{(i, i+1) | i \in \{0, 1, 2, \ldots, n-2\}\}\]. Such a graph is denoted by }P_n\textit{.}

\textbf{Exercise 1:} Verify that the following graph is }P_6\textit{. Hint: don't let the shape of the drawing sway your thoughts, use the definition.

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{path}
\caption{Path on 6 vertices}
\end{figure}

\textbf{Definition:} A graph is called a \textit{cycle on }n\textit{ vertices} (\(n \geq 3\)) if the vertices can be labeled with elements of \(\{0, 1, 2, \ldots, n-1\}\) so that the edge set is 
\[\{(i, i+1) | i \in \{0, 1, 2, \ldots, n-1\}\}\] \(\cup\) \{(0, n-1)\}. Such a graph is denoted by }C_n\textit{.}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{cycle}
\caption{\(C_3, C_4, \text{ and } C_5\)}
\end{figure}

\textbf{Definition:} A graph is called a \textit{Wheel on }n\textit{ vertices} if the vertices can be labeled with elements of \(\{0, 1, 2, \ldots, n-1\}\) so that the vertices \(\{1, 2, \ldots, n-1\}\) determine a cycle on \(n-1\) vertices and vertex 0 is adjacent to each of the vertices in \(\{1, 2, \ldots, n-1\}\) Such a graph is denoted by }W_n\textit{.}

\textbf{Exercise :} Verify that the following graph is }W_6\textit{.
Thus far we have seen some special graphs, null graphs, complete graphs, paths, cycles, and wheels. Next we consider properties of graphs. But first let us introduce a graph concept.

**Definition.** Let $G$ be a graph on vertex set $V$ and edge set $E$. A subset of the vertices, $S \subseteq V$, is called **independent** if no two of vertices of $S$ are adjacent.

**Definition.** A graph $G$ is a **bipartite graph** if the vertices can be partitioned into two sets $X$ and $Y$ such that vertices of $X$ determine an independent set, the vertices of $Y$ too determine an independent set and $X \cap Y = \emptyset$ and $X \cup Y = V(G)$. A **complete bipartite graph** is a bipartite graph in which every edge of $X$ is adjacent to every vertex of $Y$. We denote a complete bipartite graph as $K_{n,m}$, where one partition has $n$ vertices and the other has $m$ vertices.

**Example:** The graph $C_4$ is a bipartite graph since the vertex set can be partitioned as $X = \{0, 2\}$ and $Y = \{1, 3\}$, and the sets each are independent.

![Graph](image)

**Exercise:** Provide an explanation for the fact that the graph $C_3$ is not a bipartite graph.

**Exercise:** Draw an example of a bipartite graph in which each of the parts has 4 vertices (that is the graph has 8 vertices) and the number of edges is 10.

**Theorem. The Handshaking theorem.** Let $G = (V, E)$ be an undirected graph with $e$ edges. Then the sum of the degrees is equal to twice the number of edges.
Theorem. Let $G = (V, E)$ be an undirected graph has an even number of vertices of odd degree.

Definitions. A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; if $H$ is a subgraph of $G$, this is denoted by $H \subseteq G$. This notation is in some ways unfortunate since of course $H$ and $G$ are not sets, such is the situation. A subgraph $H$ of $G$ in which every edge of $G$ determined by the vertices of $H$ is in $E(H)$ is called an induced subgraph of $G$.

Examples: 
(i) Show that $C_5$ is an induced subgraph of the Peterson graph.
(ii) Verify that the Peterson graph has twelve 5-cycles.