Mathematical Induction

**Theorem.** (The Principle of Math Induction) (PMI)
Let $P(n)$ be a proposition.
If  
(i) $P(1)$ is true
(ii) If $P(k) \rightarrow P(k+1)$ for every positive integer
then $P(n)$ is true for every positive integer.

**How to use the Principle of Mathematical Induction:**
**Step 1:** Identify the math statement to be proven.
**Step 2:** Show that the statement is true for the natural number 1.
**Step 3:** Show that if we assume that the statement is true for some $k$, then it follows that the statement must also be true for $k+1$, i.e. property (ii).
**Step 4:** Conclusion: By the Principle of Math Induction….

Exercise 1: Prove the following using mathematical induction

**Theorem** $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

**Proof:** **Step 1:** Let $P(n)$ be the statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

**Step 2:**
Example 2: Prove that for all $n \in \mathbb{N}$, $\left(1 + \frac{1}{2}\right)^n \geq 1 + n/2$.

Proof: Let $P(n)$ be the statement $\left(1 + \frac{1}{2}\right)^n \geq 1 + n/2$.

Since $\left(1 + \frac{1}{2}\right)^1 = 1 + 1/2$, $P(1)$ is true.

Assume $P(k)$ is true. This means that $\left(1 + \frac{1}{2}\right)^k \geq 1 + k/2$.

Thus if $P(n)$ is true, then $P(n+1)$ is also true. Hence by PMI,

$\left(1 + \frac{1}{2}\right)^n \geq 1 + n/2$.

QED

Example 3: Can PMI be used to show that $\forall n \in \mathbb{N}$, $n = n + 1$?

Solution: Let $P(n)$ be the statement $n = n + 1$. Assume $P(k)$ is true, that is assume $k = k + 1$ for some integer $k$.

$$k + 1 = (k+1) + 1$$

since $P(k)$ is true

$$= k + 2.$$ 

Thus $P(k+1)$ is true whenever $P(k)$ is true. Hence by PMI,....?

What happened? How could we prove this nonsense?
Example 4: Sums of Geometric Progressions. Use mathematical induction to prove the following formula for the sum of a finite number of terms of a geometric progression.

\[ \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1} , \quad \text{when } r \text{ is not equal to } 1. \]

Example 5: Use mathematical induction to prove that \( 2^n < n! \) for every positive integer \( n \) with \( n \geq 4 \).
Example 6: Use mathematical induction to prove that if \( S \) is an \( n \) element set, then \( |P(S)| = 2^n \) for every positive integer \( n \).

Example 7 (Ch. 8.4 #20): Use mathematical induction to prove that if \( G = (V,E) \) is a connected graph with \( n \) vertices, then \( G \) has at least \( n-1 \) edges.