Ch. 3.2: Sequences and Summations

**Def.** Recall the natural numbers are denoted \( N = \{1, 2, 3, \ldots\} \) and the whole numbers \( W = \{0, 1, 2, 3, \ldots\} \). Let \( S \) be a subset of integers. A **sequence** is a function \( f: N \rightarrow S \) or \( f: W \rightarrow S \). Instead of \( f(n) \) we use the notation \( a_n \) to denote the image of the integer \( n \). We call \( a_n \) the \( n^{th} \) term of the sequence.

Sequences and summations occur in many Math and CS areas of study for example Linear algebra, Calculus, Number Theory, Complexity of Algorithms and Discrete Structures.

**Notation:** We use the notation \( \{a_n\} \) to describe the sequence \( a_1, a_2, a_3, \ldots \).

**Example:** Consider the sequence \( \{a_n\} \), where \( a_n = \frac{1}{n} \) *(the general formula of the sequence.)*

The list of the terms of this sequence, beginning with \( a_1 \), namely, \( a_1, a_2, a_3, a_4, \ldots \) starts with \( \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \).

**Example:** Consider the sequence \( \{b_n\} \), where \( b_n = (-1)^n \) *(also called the rule of the sequence.)*

The list of the terms of this sequence, beginning with \( b_1 \), namely, \( b_1, b_2, b_3, b_4, \ldots \) starts with \(-1, 1, -1, 1, -1, 1, \ldots\).

**Exercise:** What are the \( a_0, a_1, a_2, a_3 \) terms of the following sequences?

| \{ \( 4^n \) \} | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) |
|\{ \( \frac{n}{2} \) \} | \ | | | |
| \{ \( n! \) \} | \ | | | |
**Def.** Finite sequences $a_1, a_2, a_3, \ldots a_n$ are called **strings**. A string is also denoted by $a_1 a_2 a_3 \ldots a_n$ (without the commas.)

**Special Integer Sequences**

**Arithmetic sequences** are those such that consecutive differences are constant. The general formula for the nth term is $a + d(n-1)$ where $a$ is the first term and $d$ is the constant difference.

**Example:** 1, 3, 5, 7, 9, …
- Find the $n^{th}$ term (i.e. the general rule) _____
- The 100th term is _____

**Example:** 0, 50, 100, 150, 200, …
- Find the $n^{th}$ term _____
- The 100th term is _____

**Geometric sequences:** are those such that consecutive ratios (2nd term divided by the 1st, 3rd term divided by the 2nd, etc.) are constant. The general formula for the nth term is $ar^{n-1}$ where $a$ is the first term and $r$ is the constant ratio.

**Example:** 3, 6, 12, 24, 48, …
- Find the $n^{th}$ term _____
- Find the 100th term _____

**Example:** 10, 100, 1000, 10,000, 100,000, …
- Find the $n^{th}$ term _____
- The 100th term is _____
Suppose that we are given a numerical sequence that is neither an arithmetic nor geometric sequence. What to do then?

Other tool to consider for discovering patterns:
- Are the terms dependant on previous terms?
- Are the terms squares? Or cubes?
- Are some terms repeated?

Example: 1, 8, 27, 64, 125, …
Find a simple formula (or general rule) for the sequence.

Example: 1, 1, 2, 3, 5, 8, 13 … (Fibonacci sequence)
Find a simple formula (or general rule) for the sequence.

Summations

Notation: The sum $a_1 + a_2 + a_3 + ... + a_n$ is represented by the summation notation $\sum_{i=1}^{n} a_i$. The variable $i$ is called the index of summation.

$\sum_{i=1}^{n} a_i$ is read as "the sum for $i$ equal 1 to $n$.

$\sum_{j=m}^{n} a_j$ is read as "the sum for $j$ equal $m$ to $n$.

$n$ is called the upper limit.

$m$ is called the lower limit.

The Greek letter sigma is used to denote summation.
Exercise: Express the sum of the first 100 terms of the sequence \( \{a_n\} \), where \( a_n = \frac{1}{n} \), for \( n = 1, 2, 3, \ldots \)

Exercise: What is the value of \( \sum_{j=5}^{7} j^2 \) ?

Exercise: What is the value of \( \sum_{j=0}^{4} 2^j \) ?

Exercise: What is the value of \( \sum_{k=1}^{5} 2^{(k-1)} \) ?

Double Summations
\[
\sum_{i=1}^{3} \sum_{j=0}^{2} ij =
\]

A set index summation
\[
\sum_{s \in \{0,1,3\}} s^2 =
\]

Manipulating the Notation:
- Suppose we have the sum \( \sum_{j=1}^{7} j^2 \) and that we need to change the summation notation so that the lower limit is 0. Note that this is accomplished by letting \( m = j-1 \). Then \( j = m+1 \), and
\[
\sum_{j=1}^{7} j^2 = \sum_{m=0}^{6} (m+1)^2.
\]
- By associativity, \( \sum_{j=1}^{6} j^2 + 7^2 = \sum_{j=1}^{7} j^2 = 1 + \sum_{j=2}^{7} j^2 \)

- Note: \( \sum_{j=1}^{6} j^2 + 7^2 \neq \sum_{j=1}^{6} (j^2 + 7^2) \) (Beware of parenthesis)

- By the distributive property of multiplication over addition,

\[
3 \left( \sum_{k=1}^{n} k \right) = \sum_{k=1}^{n} 3k
\]

**Example:** The sum of the first \( n \) terms of a geometric sequence is called a **geometric progression**.

**Theorem** \( \sum_{j=0}^{n} ar^j = \frac{ar^{n+1}-a}{r-1} \) if \( r \neq 1 \)

**Proof.**