The Language of Sets

Terms: The objects in a set are called the elements, or members, of the set. A set is said to contain its elements.

Notation: We use upper case letters to denote sets, e.g. A, B, C, X,…

Def: Two sets are equal if and only if they have the same elements.
Examples: Let \( A = \{1, 2, 5\} \), \( B = \{1, 2, 3\} \), and \( C = \{5, 2, 1\} \). Sets \( A \) and \( C \) are equal (note the order does not matter), but no other combinations in this example are equal.

Notation/Definitions:
1. \( \in \) is read as "is an element of".
   Example: Let \( A = \{1, 2, 5\} \). \( 1 \in A \), \( 2 \in A \), but \( 3 \notin A \).

2. If a set is finite or has a pattern then the set can be described by listing the elements. But a more general way to describe a set is by the use of set builder notation.

Examples:
   i. \( R = \{x \mid x \text{ is a real number}\} \)
      (The braces indicate a set and the vertical bar is read as "such that")
      This is read as "the set of all \( x \) such that \( x \) is a real number"

   ii. \( O = \{x \mid x \text{ is an odd positive integer less than 10}\} \)
      This is read as "the set of all \( x \) such that \( x \) is an odd positive integer less than 10".
      (Note this set could have been listed \( O = \{1, 3, 5, 7, 9\} \).)

3. The universal set, which we will denote as \( U \), is the set of all objects under consideration in a given problem.

4. Russel’s paradox: Let \( S = \{X \mid X \notin X\} \). Analysis of this set shows that not every predicate defines a set.

5. A set \( A \) is said to be a subset of set \( B \) if and only if every element of \( A \) is also an element of \( B \).
In notation this is: \( A \subseteq B \iff \forall x (x \in A \rightarrow x \in B) \).

\( A \) is **not a subset** of \( B \) in notation: \( A \nsubseteq B \iff \exists x (x \in A \wedge x \notin B) \)

**Proof:**

<table>
<thead>
<tr>
<th>( A \nsubseteq B \iff )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg \forall x (x \in A \rightarrow x \in B) \iff )</td>
<td>By the negation of the definition of subset</td>
</tr>
<tr>
<td>( \exists x \neg (x \in A \rightarrow x \in B) \iff )</td>
<td>By the negation of the universal quantifier.</td>
</tr>
<tr>
<td>( \exists x \neg (x \in A \lor x \in B) \iff )</td>
<td>By the implication conversion law</td>
</tr>
<tr>
<td>( \exists x (x \in A \wedge x \notin B) )</td>
<td>??</td>
</tr>
</tbody>
</table>

End of Proof.

**Examples:** Let \( A = \{a, b, c\} \), \( B = \{a, b, c, d\} \), \( C = \{b, c, d\} \), and \( D = \{c, b, a\} \). Fill in the blank with the subset symbol or the not a subset of symbol.

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>( B )</td>
</tr>
<tr>
<td>( A )</td>
<td>( C )</td>
</tr>
<tr>
<td>( D )</td>
<td>( B )</td>
</tr>
<tr>
<td>( D )</td>
<td>( A )</td>
</tr>
</tbody>
</table>

5. If \( A \subseteq B \) and \( A \neq B \), then \( A \) is a **proper subset** of \( B \) denoted by \( A \subset B \).

6. \( \emptyset \) represents the **empty set** or **null set**, which is defined as the set with no elements.

**Fact:** Let the universal set be the collection of all sets.

\( \forall A, \emptyset \subseteq A. \)

**Proof:**
7. Let S be a set. If there are exactly $n$ distinct elements in S, where $n$ is a nonnegative integer, we say S is a **finite set** and $n$ is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be **infinite** if it is not finite.

**Examples:**
1. Let $A = \{1, 2, 5\}$. Then $|A| = 3$.
2. Let S be the set of letters in the English alphabet. Then $|S| = 26$.
3. $|\emptyset| = \quad 4. |\emptyset| = \quad$
6. $|\infty| = \quad 6. |\{1000\}| = \quad$
7. $|\{1,2,\ldots,1000\}| = \quad 8. |\{1,2,1000\}| = \quad$

8. Some special infinite sets and the notation that we will use

- $\mathbb{Z}$ = set of integers = $\{-\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{N} = \mathbb{Z}^+ =$ set of positive integers
- $\mathbb{Z}^- =$ set of negative integers
- $\mathbb{W} =$ set of whole numbers = $\{0, 1, 2, 3, \ldots\}$
- $\mathbb{Q} =$ set of rational numbers
- $\mathbb{R} =$ set of real numbers
- $\mathbb{R}^+ =$ set of positive real numbers
- $\mathbb{R}^- =$ set of negative real numbers

**Definition:** Given a set S, the **power set of S** is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

**Fact to be proven later:** If $|A| = n$, then $|P(A)| = 2^n$.

**Examples:**
1. Let $A = \{a, b, c\}$. Describe $P(A)$.
   **Solution:** $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Note just as describe in the fact above this power set has $2^3 = 8$ elements.
2. **You try.** Let $B = \{0, 1\}$. Describe $P(B)$.
3. $P(\emptyset) =$ ?
4. Let \( A \) be any set. True/False.

| 1. \( \emptyset \subseteq A \)     |     |
| 2. \( \{\emptyset\} \subseteq A \) |     |
| 3. \( \emptyset \subseteq P(A) \) |     |
| 4. \( \{\emptyset\} \subseteq P(A) \) |     |
| 5. \( \emptyset \in A \) |     |

5. Let \( A = \{a, b, c\} \). True/False.

| i. \( \{a\} \in A \) |     |
| ii. \( \{a\} \subseteq A \) |     |
| iii. \( \{a\} \in P(A) \) |     |
| iv. \( \{a\} \subseteq P(A) \) |     |
| v. \( \{\{a\}\} \subseteq P(A) \) |     |

**Set Operations**

**Definitions.** Let \( A \) and \( B \) be sets.

1. The **union of sets** \( A \) and \( B \) is denoted and defined as follows: \( A \cup B = \{x \mid (x \in A) \lor (x \in B)\} \)

2. The **intersection of sets** \( A \) and \( B \) is denoted and defined as follows: \( A \cap B = \{x \mid (x \in A) \land (x \in B)\} \)

3. Fact: \( |A \cup B| = |A| + |B| - |A \cap B| \)

4. Two sets \( A \) and \( B \) are said to be **disjoint** if \( A \cap B = \emptyset \).

5. The **difference of two sets** \( A \) and \( B \), denoted by \( A - B \), is defined as follows: \( A - B = \{x \mid (x \in A) \land (x \notin B)\} \).

6. The **complement of a set** \( A \), denoted by \( A' \), is defined as follows: \( A' = \{x \mid x \notin A\} \).

7. The **symmetric difference of two sets** \( A \) and \( B \), denoted
by $A \oplus B$, is defined as follows: $A - B = (A - B) \cup (B - A)$.

**Example 1:** One of the set identities that we shall use often is DeMorgan’s law for sets, which is as follows: $(A \cap B)' = A' \cup B'$.

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**Set Identities:** See the table on page 83 of our text consists of set identities, many of which we will prove.

<table>
<thead>
<tr>
<th>Set Identities</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup A = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cap A = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cup \emptyset = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cup A' = U$</td>
<td>Inverse laws</td>
</tr>
<tr>
<td>$A \cap A' = \emptyset$</td>
<td>Inverse laws</td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$(A')' = A$</td>
<td>Double complementation law</td>
</tr>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$(A \cup B)' = A' \cap B'$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$(A \cap B)' = A' \cup B'$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$A \cup (A \cap B) = A$</td>
<td>Absorption laws</td>
</tr>
<tr>
<td>$A \cap (A \cup B) = A$</td>
<td>Absorption laws</td>
</tr>
</tbody>
</table>
The following laws have no names

<table>
<thead>
<tr>
<th>Law</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $A \subseteq B$, then $A \cap B = A$.</td>
<td></td>
</tr>
<tr>
<td>If $A \subseteq B$, then $A \cup B = B$.</td>
<td></td>
</tr>
<tr>
<td>If $A \subseteq B$, then $B' \subseteq A'$.</td>
<td></td>
</tr>
<tr>
<td>If $A - B = A \cap B'$.</td>
<td></td>
</tr>
<tr>
<td>$A \oplus B = (A \cup B) - (A \cap B)$</td>
<td></td>
</tr>
</tbody>
</table>

There are many proof techniques used to prove set identities (we will omit membership tables.) Two of these methods are illustrated in what follows.

**Example 2:** Prove DeMorgan’s Law: $(A \cup B)' = A' \cap B'$.

**Proof:**

\[
(A \cup B)' = \\
\{ x \mid x \notin (A \cup B) \} = \text{By definition of the complement of a set.} \\
\{ x \mid \neg (x \in (A \cup B)) \} = \text{Symbolic notation of the negation.} \\
\{ x \mid \neg (x \in A \lor x \in B) \} = \text{By definition of the union} \\
\{ x \mid \neg (x \in A) \land \neg (x \in B) \} = ??? \\
\{ x \mid (x \notin A) \land (x \notin B) \} = \text{From symbolic notation to set notation.} \\
\{ x \mid (x \in A') \land (x \in B') \} = ??? \\
A' \cap B' = \text{By ???} \\
\]

Hence, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

End of Proof.

**Example 3:** Prove $A \cup U = U$ (Note that this is one of the domination laws)

**Example 4:**

True or False: The union operator is commutative.
True or False: The difference operator is commutative.
True or False: The set difference operator is commutative.
**Example 5**: Show \( \overline{A \cup (B \cap C)} = (C \cup \overline{B}) \cap \overline{A} \)

Proof.

<table>
<thead>
<tr>
<th>( \overline{A \cup (B \cap C)} )</th>
<th>( \overline{A} \cap (B \cap C) )</th>
<th>By DeMorgan's Law for sets.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = \overline{A} \cap \overline{B} \cup \overline{C} )</td>
<td>( = (B \cup \overline{C}) \cap \overline{A} )</td>
<td>By DeMorgan's Law for sets.</td>
</tr>
<tr>
<td></td>
<td>( = (C \cup \overline{B}) \cap \overline{A} )</td>
<td>??</td>
</tr>
</tbody>
</table>

End of Proof.

**Example 6**: Find the errors in the following proof. Note the statement is true but the proof has errors.

Prove that \( A - B = A \cap \overline{B} \)

Proof.

| \( A - B \) | \( = \{x | (x \in A) \cap (x \notin B)\} \) | By definition of set difference |
|-------------|--------------------------------|---------------------------------|
|             | \( = x | (x \in A) \cap (x \in \overline{B}) \) | By definition of the complement of a set |
|             | \( = x | A \cap \overline{B} \) | By definition of the intersection |

End of Proof.

**Generalized Unions and Intersections**

Let \( A_1, A_2, \ldots, A_n \) be sets.

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n.
\]

**Note**: an element is in the union if it is in at least one of the \( A_i \) for \( i=1, 2, 3, \ldots n \).

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n.
\]

**Note**: an element is in the intersection if it is in each of the \( A_i \) for \( i=1, 2, 3, \ldots n \).
Example 7.

Let \( A_i = \{1,2,3,...,i\} \) for \( i = 1,2,3,... \)

(a) Find \( \bigcup_{i=1}^{n} A_i = \)

(b) Find \( \bigcap_{i=1}^{n} A_i = \)

Example 8.

Let \( A_i = \{i,i+1,i+2\} \) for \( i = 1,2,3,... \)

(a) Find \( \bigcup_{i=2}^{n} A_i = \)

(b) Find \( \bigcap_{i=2}^{n} A_i = \)

**Definition:** The **ordered n-tuple** \((a_1, a_2, a_3,...,a_n)\) is the ordered collection that has \(a_1\) as its first element, \(a_2\) as its second element, ..., and \(a_n\) as its \(n^{th}\) element.

**Definition:** \((a_1, a_2, a_3,...,a_n) = (b_1, b_2, b_3,..., b_n)\) if and only if 
\( (a_1 = b_1) \land (a_2 = b_2) \land .... \land (a_n = b_n) \)

**Terminology:** \((a_1, a_2)\) is called an **ordered pair**.
\((a_1, a_2, a_3)\) is called an **ordered triple**.

**Definition:** Let \( A \) and \( B \) be sets. The **Cartesian Product of A and B**, denoted \( A \times B \), is the set of all ordered pairs \((a,b)\) where \( a \in A \) and \( b \in B \).

Hence in set builder notation, the cartesian product is 
\( A \times B = \{(a,b) \mid a \in A \land b \in B\} \).

**Example 9:** Let \( A = \{a, b\} \) and \( B = \{0, 1, 2\} \). Find \( A \times B \) and \( B \times A \).
Def: **Generalized Cartesian Product**: Let $A_1, A_2, ..., A_n$ be sets. 

\[ A_1 \times A_2 \times ... \times A_n = \{ (a_1, a_2, ..., a_n) \mid (a_1 \in A_1) \land (a_2 \in A_2) \land ... \land (a_n \in A_n) \} \]

**Definition**: Let $I$ be an index set and $P$ a family of subsets $S_i$ of a nonempty set $S$, where $i \in I$. Then $P$ is a **partition** of $S$ if

- Each $S_i$ is nonempty
- The subsets are pairwise disjoint, that is, $S_i \cap S_j = \emptyset$ if $i \neq j$.
- The union of the subsets $S_i$ is $S$, that is $\bigcup_{i \in I} S_i = S$.

(Each subset $S_i$ is a **block** of the partition.)

**Example 10**: Let $Z$ denote the set of integers which, when divided by 5, leave $r$ as the remainder. The $0 \leq r < 5$. Let

\[
Z_0 = \{ ..., -5, 0, 5, ... \}
\]
\[
Z_1 = \{ ..., -4, 1, 6, ... \}
\]
\[
Z_2 = \{ ..., -3, 2, 7, ... \}
\]
\[
Z_3 = \{ ..., -2, 3, 8, ... \}
\]
\[
Z_4 = \{ ..., -1, 4, 9, ... \}.
\]

Then $P = \{ Z_0, Z_1, Z_2, Z_3, Z_4 \}$ is a partition of the set of integers.