Ch 1.3 Quantifiers

Recall that the expression \( x > 3 \) is not a proposition. Why?

**\( P(x) \) Notation:**
- We will use the propositional function (or predicate) notation \( P(x) \) to denote the expression " \( x \) has property \( P \)".
- As soon as \( x \) is assume to be some value then \( P(x) \) has a truth value.

The goal of the section is to quantify such statements so that the result is a proposition. The process is very much in the spirit of the next example.

**Example:** Let \( P(x) \) denote the statement " \( x > 3 \)".
What are the truth values:

i. \( P(4) \) True or False ?
ii. \( P(2) \) True or False ?
iii. \( P(y) \) True or False ?

**Example:** Let \( Q(x,y) \) denote the statement " \( x = y + 3 \)".
What are the truth values:

i. \( Q(7,4) \) True or False ?
ii. \( Q(2,2) \) True or False ?

**Quantifiers**

*Def.* The **universe of discourse** for a math statement is the domain of that statement.

*Def.* The **universal quantification of \( P(x) \)** is the proposition
" \( P(x) \) is true for all values of \( x \) in the universe of discourse." 

- This proposition is denoted by \( \forall x \ P(x) \).
- The proposition \( \forall x \ P(x) \) is read as "for all \( x \) \( P(x) \)" or "for every \( x \) \( P(x) \)".
- The symbol \( \forall \) is called the **universal quantifier**.

*Def.* The **existential quantification of \( P(x) \)** is the proposition
" There exists an element \( x \) in the universe of discourse for which \( P(x) \) is true."

- This proposition is denoted by \( \exists x \ P(x) \).
- The symbol \( \exists \) is called the **existential quantifier**.
- The proposition \( \exists x \ P(x) \) is read as "for some \( x \) \( P(x) \)" or "there exists an \( x \) such that \( P(x) \)".
The Truth Values of $\forall x P(x)$ and $\exists x P(x)$

**Example 1:** Suppose that the universe of discourse for $x$ is the set of nonnegative integers \{0, 1, 2, 3, 4, 5, …\}

What is the truth value of $\forall x (x > 0)$? ________

What is the truth value of $\exists x (x > 0)$? ________

**Example 2:** Suppose that the universe of discourse for $P(x)$ is the small set \{ $x_1$, $x_2$, $x_3$, $x_4$ \}.

Since $\forall x P(x) \iff P(x_1) \land P(x_2) \land P(x_3) \land P(x_4)$

when is $\forall x P(x)$ true? ___________________________________

when is $\forall x P(x)$ false? ___________________________________

Since $\exists x P(x) \iff P(x_1) \lor P(x_2) \lor P(x_3) \lor P(x_4)$

when is $\exists x P(x)$ true? ___________________________________

when is $\exists x P(x)$ false? ___________________________________

**Note:** The above observations also hold for infinite universe of discourse sets.

**Examples 3:**

<table>
<thead>
<tr>
<th>Where $P(x)$ denotes:</th>
<th>Universe of Discourse for $P(x)$</th>
<th>Truth value of $\exists x P(x)$</th>
<th>Truth value of $\forall x P(x)$</th>
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</thead>
<tbody>
<tr>
<td>$X = x + 3$</td>
<td>Real numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X = x + 0$</td>
<td>Real numbers</td>
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<td>$0/x = 0$</td>
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<td>$0 - x = x$</td>
<td>Natural numbers</td>
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</tr>
</tbody>
</table>

**Example 4:**

(i) What is the truth value of $[\forall x P(x) \rightarrow \exists x P(x)]$? __________

(ii) What is the truth value of $[\exists x P(x) \rightarrow \forall x P(x)]$? __________

**Example 5:** Let $P(x,y)$ denote the statement "$x^2 = y$". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is the English equivalent to $\exists y \forall x P(x,y)$

(a) For every real number $y$ there is a real number $x$ such that $x^2 = y$.

(b) For every real number $y$ and for every real number $x$ it is the case that $x^2 = y$. 
(c) There is a real number \( y \) such that for every real number \( x \) it is the case that \( x^2 = y \).

- What is the truth value of \( \exists y \forall x P(x,y) \)?

**Example 6:** Let \( P(x,y) \) denote the statement " \( x^2 = y \) ". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is an English equivalent to \( \forall y \exists x P(x,y) \)?
  (a) For every real number \( y \) there is a real number \( x \) such that \( x^2 = y \).
  (b) For every real number \( y \) and for every real number \( x \) it is the case that \( x^2 = y \).
  (c) There is a real number \( y \) such that for every real number \( x \) it is the case that \( x^2 = y \).

- What is the truth value of \( \forall y \exists x P(x,y) \)?

**Definition.** A real number \( x \) is a **rational number** if can be expressed as the ratio of two integers \( a \) and \( b \) with \( b \) not zero, that is if \( x = \frac{a}{b} \), with \( b \neq 0 \).

**Example 7:** Let \( Q(x) \) denote the statement " \( x \) is a rational number ", and \( I(y) \) denote the statement " \( y \) is an integer. Assume the universe of discourse is the set of real numbers.

i. \( \exists y (Q(y) \land I(y)) \)

- Which of the following is an English translation?
  (a) There is a rational number \( y \) which is a rational number and also an integer.
  (b) There is a real number \( y \) which is a rational number and also an integer.
  (c) Every real number \( y \) is a rational and also an integer.

- What is the truth value? 

ii. \( \forall y (Q(y) \land I(y)) \)

- Which of the following is an English translation?
  (a) There is a rational number \( y \) which is a rational number and also an integer.
  (b) There is a real number \( y \) which is a rational number and also an integer.
  (c) Every real number \( y \) is a rational or it is an integer.
  (d) Every real number \( y \) is a rational and also an integer.

- What is the truth value? 

iii. $\forall y (I(y) \rightarrow Q(y))$

- Which of the following is the English translation?
  (a) Every real number $y$ is an integer and also a rational number.
  (b) There is a real number $y$ such that if $y$ is an integer then $y$ is a rational number.
  (c) For any real number $y$, if $y$ is an integer then $y$ is a rational number.

- What is the truth value? ___________

**Negations of Quantified Expressions**

**Example 8**: Consider the proposition "Every student in the class has taken a course in calculus."

- When is this proposition true? _________________

- Write the proposition in symbolic logic notation.
  ____________________________________________
  ____________________________________________

- Which of the following is the negation of "Every student in the class has taken a course in calculus."
  (a) No student in the class has taken calculus.
  (b) Some student in the class has taken calculus
  (c) Some student in the class has not taken calculus

- Write the negation of the proposition in symbolic logic notation.
  ____________________________________________
  ____________________________________________

- When is the negation of the proposition "Every student in the class has taken a course in calculus" true? _________________

**Fact:**

\[ \neg \forall x P(x) \iff \exists x \neg P(x) \]

\[ \neg \exists x P(x) \iff \forall x \neg P(x) \]

**Example 9:**

(iii) What is the truth value of $[\forall x P(x)] \lor [\neg \forall x P(x)]$? ____________

(iv) What is the truth value of $[\forall x P(x)] \lor [\exists x \neg P(x)]$? ____________

(v) What is the truth value of $[\forall x P(x)] \land [\exists x \neg P(x)]$? ____________