Test 3 2405

Name __________________

I. Graph Theory

1. (13 pts) True/False.

a) **True** or **False**: There exists a graph on 4 vertices and 6 edges.

b) **True** or **False**: There exists a graph on 4 vertices and 8 edges.

c) **True** or **False**: There exists a graph on 5 vertices with degree sequence 3, 3, 3, 2, 2.

d) **True** or **False**: There exists a graph on 6 vertices with degree sequence 1, 1, 1, 1, 1, 1.

e) **True** or **False**: If two graphs are isomorphic, then the two graphs have the same degree sequence.

f) **True** or **False**: If two graphs have the same degree sequence, then the two graphs are isomorphic.

g) **True** or **False**: A cycle on $n$ vertices has $n$ edges (assuming $n$ is at least 3).

h) **True** or **False**: A path on $n$ vertices has $n - 1$ edges.

i) **True** or **False**: A complete graph on $n$ vertices has $n(n - 1)$ edges.

j) **True** or **False**: The maximum degree of the complement of a cycle on 8 vertices is 6.

k) **True** or **False**: The maximum degree of the complement of a path on 8 vertices is 6.

l) **True** or **False**: The cycle graph on 5 vertices is also a bipartite graph.

m) **True** or **False**: For any graph $G$ the number of edges is the sum of the degrees.

\[
\sum_{v \in V(G)} \deg v = 2 \cdot \text{number of edges}
\]
2. (8 pts) Draw the graph defined by on vertex set \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} such that two vertices \(u\) and \(v\) are adjacent if and only if \(u = v \pmod{4}\).

![Graph image]

3. (6 pts) Prove or disprove that following two graphs are isomorphic.

An isomorphism (function) would need to map:
- \(b\) to 1 and \(d\) to 6 or \(b\) to 6 and \(d\) to 1 because they are the only degree 3 vertices but in either case the function could not be edge preserving since \(1\) to \(a\) but \(b\) to \(d\).

4. (6 pts) Prove or disprove that following two graphs are isomorphic.

The two graphs have different number of vertices, so no function from \(G_1\) to \(H\) can be onto \(H\), and no function from \(H\) to \(G_1\) can be one-to-one. Thus \(G_1 \not\cong H\).
Integers and Division

5. (8 pts)
   i. Find GCD($2^4 \cdot 5^4 \cdot 7^3 \cdot 17$, $2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$)

   $\boxed{2^4 \cdot 5^3 \cdot 7^2 \cdot 17}$

   ii. Find LCM($2^4 \cdot 5^4 \cdot 7^3 \cdot 17$, $2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$)

   $\boxed{2^4 \cdot 5^4 \cdot 7^3 \cdot 17^3 \cdot 19}$

   iii. If the product of two integers $a \cdot b = 2^3 \cdot 3^4 \cdot 5^7 \cdot 7$ and the GCD of $a$ and $b$ is $2^2 \cdot 3 \cdot 5^6$, then what is the LCM of $a$ and $b$.

   \[
   \text{Since } \quad a \cdot b = \text{gcd}(a,b) \cdot \text{lcm}(a,b) \\
   \text{lcm} = \frac{a \cdot b}{\text{gcd}(a,b)} = \frac{2^3 \cdot 3^4 \cdot 5^7 \cdot 7}{2^2 \cdot 3 \cdot 5^6} = \boxed{2^1 \cdot 3^3 \cdot 5 \cdot 7^2}
   \]

6. (6 pts) Demonstrate the Euclidean Algorithm for finding the GCD of 2024 and 1024. (Note that the GCD is not enough, you must demonstrate the Euclidean algorithm).

   2024 = 1 \cdot 1024 + 1000
   1024 = 1 \cdot 1000 + 24
   1000 = 41 \cdot 24 + 16
   \text{gcd: } 24 = 1 \cdot 16 + 8
   16 = 2 \cdot 8 + 0

7. (10 pts) True/False.
   a. \(\boxed{\text{True or False}}\): The integers have the closure property with respect to subtraction.
   b. \(\boxed{\text{True or False}}\): The sum of two consecutive integers is odd.
   c. \(\boxed{\text{True or False}}\): The product of two consecutive integers is odd.
   d. \(\boxed{\text{True or False}}\): 8 is a divisor of 64
   e. \(\boxed{\text{True or False}}\): 64 is a factor of 8
   f. \(\boxed{\text{True or False}}\): For $a$ and $b$ nonzero integers, $(a \mid b \land b \mid c) \rightarrow a \mid c$
   g. \(\boxed{\text{True or False}}\): For a nonzero integer $a$, $a \mid (b + c) \rightarrow a \mid c$
   h. \(\boxed{\text{True or False}}\): $12 \mod 4 = 0$
   i. \(\boxed{\text{True or False}}\): $-4 \equiv 12 \mod 4$
   j. \(\boxed{\text{True or False}}\): If $a \equiv b \mod m$ and $c \equiv d \mod m$, then $a + b \equiv c + d \mod m$
8. (9 points) Store the following student ID numbers in memory locations 0-9 using a hashing function $f(s) = s \mod 10$ (assume they are processed in the order given here). Explain how you managed collisions.

658779, 658772, 657102, 648559, 657991, 657122, 658881, 648557, 657100

<table>
<thead>
<tr>
<th>Memory location</th>
<th>ID numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>648559</td>
</tr>
<tr>
<td>1</td>
<td>657991</td>
</tr>
<tr>
<td>2</td>
<td>658772</td>
</tr>
<tr>
<td>3</td>
<td>657102</td>
</tr>
<tr>
<td>4</td>
<td>657122</td>
</tr>
<tr>
<td>5</td>
<td>658881</td>
</tr>
<tr>
<td>6</td>
<td>657100</td>
</tr>
<tr>
<td>7</td>
<td>648557</td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>658779</td>
</tr>
</tbody>
</table>

9. (8 pts) Let $a$, $b$ and $c$ be integers such that $c$ is nonzero. Complete the proof of \[\text{if } c \mid a \text{ and } c \mid b, \text{ then } c \mid (xa + yb) \text{ for any integers } x \text{ and } y.\]

**Proof:** Let $a$, $b$ and $c$ be integers such that $c$ is nonzero. Assume that $c \mid a$ and $c \mid b$. Then by \textbf{def of divisibility}, $a = c k$ for some integer $k$ and $b = c j$ for some integer $j$. Let $x$ and $y$ be any integers. Then by \textbf{substitution},

\[xa + yb = x(ck) + y(cj) = c(xk + yj) \quad \text{by } \textbf{arithmetic}.\]

Since $x$, $k$, $y$ and $j$ are integers and the integers have closure with respect to multiplication and addition, $xk + yj$ is an integer. Thus, by definition of divisibility, $c \mid (xa + yb)$. Hence, if $c \mid a$ and $c \mid b$, then $c \mid (xa + yb)$ for any integers $x$ and $y$. \qed
Mathematical Induction

10. (10 pts) Use the principal of math induction to prove the following.

\[ \forall n \in \mathbb{N}, \sum_{i=0}^{n} (4i+1) = (n+1)(2n+1) \]

**Proof:**

Since \( \sum_{i=0}^{0} (4i+1) = 0 + 1 = (0+1)(2 \cdot 0 + 1) \),
our basis step is established.

Assume \( \sum_{i=0}^{k} (4i+1) = (k+1)(2k+1) \) for some \( k \in \mathbb{N} \).

By associativity,

\[
\sum_{i=0}^{k+1} (4i+1) = \sum_{i=0}^{k} (4i+1) + 4(k+1) + 1
\]

\[
= (k+1)(2k+1) + 4(k+1) + 1 \text{ by inductive assumption}
\]

\[
= 2k^2 + 7k + 6
\]

\[
= (2(k+1)^2) + (2(k+1) + 1) \]

Thus, by the PMI,

\[ \forall n \in \mathbb{N}, \sum_{i=0}^{n} (4i+1) = (n+1)(2n+1) \]

Counting

11. (6 pts) For \( A = \{1, 2, 3, 4, 5, 6, 7\} \), determine the following:

i. The number of all subsets of \( A \).

\[ 2^7 \]

ii. The number of functions from \( A \) to \( A \).

\[ 7^7 \]

iii. The number of one-to-one functions from \( A \) to \( A \).

\[ 7! \]
12. (3 pts) Determine the number of bit strings of length 10 that contain at most one zero bit.

(a) 10 (b) $2^{10}$ (c) $2^{10} - 1$ (d) 11 (e) none of these

- 10 with 1 zero bit
- 10 without any 0

13. (6 pts) Find the number of positive integers less than or equal to 2000 that are divisible by 7.

\[ \left\lfloor \frac{2000}{7} \right\rfloor \]

b. that are divisible by 5 but not 7.

\[ \left\lfloor \frac{2000}{5} \right\rfloor - \left\lfloor \frac{2000}{35} \right\rfloor \]

14. (3 points) Among a group of 165 students, 80 are taking Cal I and 95 are taking CS I. How many are taking both of these courses?

Let

\[ A \text{ be students in Cal I} \]
\[ B \text{ be students in CS I} \]

Question \( |A \cap B| = ? \)

Since \( |A \cup B| = |A| + |B| - |A \cap B| \)
\[ |A \cap B| = |A| + |B| - |A \cup B| \]
\[ = 80 + 95 - 165 \]
\[ = 10 \]