Permutations and Combinations

- What do we mean by a permutation of the elements of a set?

- There are a couple of ways to define this concept and we will look at two of them.

But first let us look at an example: Let us consider the elements $a$, $b$, and $c$ as follows:

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Each is a rearrangement of the three objects $a$, $b$ and $c$.

Another way to look at each of the rearrangements is as a one-to-one function. For example:

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<thead>
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<th>1</th>
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<th>3</th>
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<td>$f_1$</td>
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<td>$f_6$</td>
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**Question:** How many rearrangements are there of an $n$-element set? (i.e. how many one-to-one functions are there from an $n$-element set to another $n$-element set?)

More generally, we are interested in the following types of permutations. Given $n$ objects how many $r$-element rearrangements are there of these objects assuming that $n > r$. 
Example: Given the 26 letters in the alphabet, how many 3-letter arrangements are possible, assume no letter is repeated. Some examples of 3-letter arrangements are \((a, b, c), (a, f, k), (p, b, n)\) and so on.

Solution: Each 3-letter arrangement is a one-to-one function. For example: \((a, f, k)\) is the function \(f : \{1, 2, 3\} \rightarrow \{a, b, c, \ldots, y, z\}\), define as \(f(1) = a, f(2) = f, \) and \(f(3) = k\).

Thus there are ______________ 3-letter arrangements of the alphabet.

Thus on the one hand: An **k-permutation** of an \(m\)-element set with distinct elements, \(S\), is a one-to-one function \(f : \{1, 2, \ldots, k\} \rightarrow S\).

Another definition: A **k-permutation** of an \(m\)-element set with distinct elements is an ordered arrangement of \(k\) elements of the set.

**Theorem.** The number of \(k\)-permutations of a set with \(m\) distinct elements, where \(0 \leq k \leq m\), is

\[
P(m,k) = (m)_k = \frac{m!}{(m-k)!}
\]

**Proof.** Each \(k\)-permutation is a one-to-one function from a \(k\)-element set to an \(m\)-element set. Thus by Example from previous note (discussed in class), the result follows.

**Def.** A **k-combination** of elements of an \(m\)-element set is an unordered selection of \(k\) elements from the set, i.e. an \(r\)-combination is simply a subset of the set with \(r\) elements.

The number of \(k\)-permutations of a set with \(m\) distinct elements is **not equal** to the number of \(k\)-combination of elements of an \(m\)-element set, but they are related.
Theorem. The number of $k$-combinations of a set with $m$ distinct elements, where $0 \leq k \leq m$, is

$$C(m,k) = \binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

Proof: Let $C(m,k) = n$. The $k$-permutations of a set with $m$ distinct elements can be obtained by forming the $n$ $r$-combinations of the set, and then ordering the elements in each $r$-combinations, which can be done in $P(k,k)$ ways. Consequently,

$$P(m,k) = C(m,k) P(k,k).$$

This implies that

$$C(m,k) = \frac{P(m,k)}{P(k,k)} = \frac{m!(m-k)!}{k!(k-1)!} = \frac{m!}{k!(m-k)!}.$$

Examples:

i) How many subsets of the alphabet are there?

ii) How many subsets of the alphabet contain exactly 3 letters?

iii) How many subsets of the alphabet contain exactly 5 letters?

iv) How many subsets of the alphabet contain at most 5 letters?

v) How many subsets of the alphabet contain at more than 5 letters?

Example: In how many ways can two students be chosen from a class of 18?
Examples:
   i) how many permutations of the letters abcdefgh are there?
   ii) how many permutations of the letters abcdefgh contain the string dfg?

Example: In how many ways can two students be chosen from a class of 18 if one of them receives an A and the other receives a B?

Example: How many ways are there to select 5 players from a 10-member tennis team to make a trip to another school?

Example: How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Corollary: Let $m$ and $k$ be nonnegative integers with $k \leq m$. Then \[ \binom{m}{k} = \binom{m}{m-k}. \]

Proof: