Introductory Graph Theory

1. Definition of a Graph

Intuitive Definition: A simple graph is a collection of vertices (visualized as dots) and edges (visualized as arcs between dots).

Figure 1.1

Formal Definition: A simple graph $G$ with $n$ vertices and $m$ edges consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $E(G)$ is an unordered pair of vertices.

Definitions: A vertex $v$ is incident to an edge if $v$ is one of the pair of vertices which determines the edge. The degree of a vertex is the number of edges to which it is incident. We denote the degree of a vertex $v$ as $\text{deg}(v)$. Given a graph $G$ on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the degree sequence of $G$ is $\text{deg}(v_1), \text{deg}(v_2), \ldots, \text{deg}(v_n)$. The maximum value of the degree sequence is the maximum degree of the graph and we denote it by $\Delta(G)$. The minimum value of the degree sequence is the minimum degree of the graph and we denote it by $\delta(G)$.

2. Some Simple Applications of Graphs

**Acquaintance Graph.** Suppose that the people are vertices and that there is an edge between two people if they are acquaintances.

**Exercise:** Determine the following.
- The number of vertices is ______
- The maximum degree is ______
- The minimum degree is ______

Interpret the maximum and minimum degree.
Network Graph. Suppose that the computers are vertices and that there is an edge between two vertices if they have direct communication. **Exercise:** Determine the following.
The number of vertices is _____
The maximum degree is ______
The minimum degree is __
Interpret the maximum and minimum degree.

3. Some Special Graphs and More Graph Concepts and Invariants

A **graph concept** is a property of the graph. A **graph invariant** is a numeric value associated with the graph, usually independent of the way the graph is drawn.

**Definition:** Let $G$ be a graph on vertex set $V$ and edge set $E$. For any two vertices, say $u$ and $v$, if $(u, v)$ is an edge we say that $u$ is adjacent to $v$. To indicate that two vertices $u$ and $v$ are adjacent we use the notation $u \sim v$.

**Example:** Since $(a, b)$ is an edge $a \sim b$.

**Definition:** The **empty graph on $n$ vertices** (also called the null graph on $n$ vertices) is the graph on $n$ vertices with no edges.

**Definition:** The **complete graph on $n$ vertices** is the graph on $n$ vertices in which every two vertices are adjacent. We use the notation $K_n$ to denote the complete graph on $n$ vertices.
**Definition:** Let $G$ be a graph on vertex set $V$ and edge set $E$. We define the **complement graph of $G$**, denoted $\overline{G}$, as a graph on the the same vertex set $V$ in which two vertices adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

It is easily seen that the empty graph on $n$ vertices is denoted by $K_n$.

**Exercise:** Draw $\overline{G}$ if $G$ is the graph drawn below.

**Exercise:** Find a graph (on four vertices) for which $G$ and $\overline{G}$ can be drawn so they appear the same; formally this should be so that $G \simeq \overline{G}$. The symbol $\simeq$ is used to indicate that the two graphs are essentially the same except for the way the vertices are labeled (formally the symbol $\simeq$ is read as *isomorphic to*, which we will investigate soon.)

**Exercise:** Suppose we are given a graph on vertex set $\{0, 1, 2, 3, 4\}$ and that edges of this graph are determined by the following rule: $\forall x, y \in \{0, 1, 2, 3, 4\} \ x \sim y \iff |x - y| = 1$. Draw the edges subject to the given rule in the following diagram.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

**Definition:** A graph is called a **Path on $n$ vertices** if the vertices can be labeled with elements of $\{0, 1, 2, ..., n-1\}$ so that the edge set is $\{(i, i + 1) \mid i \in \{0, 1, 2, ..., n-1\}\}$. Such a graph is denoted by $P_n$.

**Exercise:** Verify that this graph in the figure is $P_3$. 
**Definition:** A graph is called a cycle on \( n \) vertices (\( n \geq 3 \)) if the vertices can be labeled with elements of \( \{0, 1, 2, \ldots, n-1\} \) so that the edge set is \( \{(i, i+1) | i \in \{0, 1, 2, \ldots, n-1\}\} \cup \{(0, n-1)\} \). Such a graph is denoted by \( C_n \).

![Figure 1.3. \( C_3 \), \( C_4 \), and \( C_5 \)](image)

**Definition:** A graph is called a Wheel on \( n \) vertices if the vertices can be labeled with elements of \( \{0, 1, 2, \ldots, n-1\} \) so that the vertices \( \{1, 2, \ldots, n-1\} \) determine a cycle on \( n-1 \) vertices and vertex 0 is adjacent to each of the vertices in \( \{1, 2, \ldots, n-1\} \). Such a graph is denoted by \( W_n \).

**Exercise:** Verify that the following graph is \( W_6 \).

**Definition.** The n-Cube graph, denoted \( Q_n \), is the graph that has vertices representing the \( 2^n \) bit strings of length \( n \). Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one position.

**Exercise:** Draw the edges of the 3-cube in the next figure.