Math 3312 Set Theory Spring 2008
Introduction

Set theory is a branch of mathematics that deals with the properties of well-defined collections of objects, which may or may not be of a mathematical nature, such as numbers or functions. The theory is less valuable in direct application to ordinary experience than as a basis for precise and adaptable terminology for the definition of complex and sophisticated mathematical concepts.

Between the years 1874 and 1897, the German mathematician and logician Georg Cantor created a theory of abstract sets of entities and made it into a mathematical discipline. This theory grew out of his investigations of some concrete problems regarding certain types of infinite sets of real numbers. A set, wrote Cantor, is a collection of definite, distinguishable objects of perception or thought conceived as a whole. The objects are called elements or members of the set.

The theory had the revolutionary aspect of treating infinite sets as mathematical objects that are on an equal footing with those that can be constructed in a finite number of steps. Since antiquity, a majority of mathematicians had carefully avoided the introduction into their arguments of the actual infinite (i.e., of sets containing an infinity of objects conceived as existing simultaneously, at least in thought). Since this attitude persisted until almost the end of the 19th century, Cantor's work was the subject of much criticism to the effect that it dealt with fictions—indeed, that it encroached on the domain of philosophers and violated the principles of religion. Once applications to analysis began to be found, however, attitudes began to change, and by the 1890s Cantor's ideas and results were gaining acceptance. By 1900, set theory was recognized as a distinct branch of mathematics.

At just that time, however, several contradictions in so-called naive set theory were discovered. In order to eliminate such problems, an axiomatic basis was developed for the theory of sets analogous to that developed for elementary geometry. The degree of success that has been achieved in this development, as well as the present stature of set theory, has been well expressed in the Nicolas Bourbaki, Elements de mathematique (begun 1939; "Elements of Mathematics"): "Nowadays it is known to be possible, logically speaking, to derive practically the whole of known mathematics from a single source, The Theory of Sets."

Source: Encyclopædia Britannica Online.
http://www.britannica.com/eb/article-9109532/set-theory#24027.toc

Naïve Set Theory

1.1 Set Notation and Some Definitions

Terms: A **set** (at this point in our course) will be thought of as a collection of objects whose elements are distinguishable. The **objects** in a set are called the **elements**, or **members**, of the set. A set is said to contain its elements.

Conventions/Notation/Definitions:

- We try to use lower case letters for elements and upper case letters to denote sets, e.g. $A, B, C, X, \ldots$
- $\in$ is read as "**is an element of**".
  Example: Let A = $\{1, 2, 5\}$. $1 \in A$, $2 \in A$, but $3 \notin A$.
- If a set is finite or has a pattern then the set can be described by listing the elements.
- A more general way to describe a set is by the use of **set builder notation**. Let $P(x)$ be a formula about x or a property of x. Then we use $\{ x \mid P(x) \}$ we denote the collection of elements x that satisfy $P(x)$.
  Examples:
  i. $R = \{x \mid x$ is a real number $\}$
  This is read as " the set of all x such that x is a real number "

  ii. $O = \{x \mid x = 2k + 1$ for some integer $k\}$
  (This set could have been listed $O = \{\ldots -3, -1, 1, 3, 5 \ldots \}$.*)*

- **Two sets A and B are equal** denoted $A = B$ if and only if they have the same elements.

Note: One example of why there is issue with the vague definition of a set. Consider $R = \{X \mid X \notin X \}$. What is the truth value of $R \in R$ and $R \notin R$?

Now more definitions and notation.

- The **universal set**, which we will denote as U, is the set of all objects under consideration in a given problem sometimes referred to as a super-set.
- **A set A is said to be a subset of set B** if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to write that A is a subset of B (note some books use $A \subset B$. To indicate that A is not a subset of B we may write $A \not\subset B$.

Theorem 1.1 (Properties of $\subseteq$)

1. $A \subseteq A$ (reflexive property)
2. if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (transitive property)
3. $A \subseteq B$ and $B \subseteq A$ if and only if $A = B$ (antisymmetric property)

- $\varnothing$ represents the **empty set** or **null set**, which is defined as the set with no elements.

Theorem 1.2 The empty set is a subset of any set.

Corollary 1.3. A nonempty set has at least two subsets.

Corollary 1.4. There is at most one empty set.

Example: The number of 2 element subsets of a set with n elements is denoted by $\binom{n}{2} = \frac{n(n-1)}{2}$

Let us examine this by example first. Let $A = \{a, b, c\}$. The collection of all 2-element sets is { {*a*,*b*}, {*a*,*c*}, {*b*,*c*} }, and $\binom{3}{2} = \frac{3(3-1)}{2} = 3$.

Theorem 1.5: The number of k-element subsets of a set with n elements is denoted by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Theorem 1.5: If n is a nonnegative integer and $k \le n$, $\binom{n}{k} = \binom{n}{n-k}$

- Given a set S, the **power set of S** is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Examples:

1. Let $A = \{a, b, c\}$. Describe the set $\mathcal{P}(A)$.
   Solution: $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

2. $\mathcal{P}(\varnothing) = ?$

3. Let A be any set. True/False.

1.	$\varnothing \subseteq A$	
2.	$\{\varnothing\} \subseteq A$	
3.	$\varnothing \subseteq P(A)$	
4.	$\{\varnothing\} \subseteq P(A)$	
5.	$\varnothing \in A$	

Theorem: The number of *all* subsets of a set with n elements is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a **finite set** and n is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be **infinite** if it is not finite.

In our new notation

Theorem: If $|A| = n$, then $|\mathcal{P}(A)| = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

Examples

1. Let $A = \{1, 2, 5\}$. Then $|A| = 3$.
2. Let S be the set of letters in the English alphabet. Then $|S| = 26$.
3. $|\varnothing| =$ _______
4. $|\{1000\}| =$ ______
5. $|\{1,2,\ldots,1000\}| =$ ______
6. $|\{\varnothing\}| =$ ________
7. $|\{\infty\}| =$ _______
8. $|\{1,2,1000\} =$ _______

Later in the course we will discuss cardinality more formally and for infinite sets.

Homework:

1. Prove the transitive property of $\subseteq$. (Part 2 of Theorem 1.1)
2. Explain why $\varnothing \neq \{ \varnothing \}$
3. Give an example of a set X such that X has an element that is also a subset of X.
4. List the elements of $\mathcal{P}(\varnothing)$
5. How many elements does $\mathcal{P}(\mathcal{P}(\varnothing))$ have ? list the elements.
6. How many elements does $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))$ have ?
7. How many 2-elements subsets does $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))$ have ?
8. Prove $\{\{x\} , \{x,y\}\} = \{\{z\} , \{z, t\}\}$ if and only if $x = z$ and $y = t$.