Countable Sets

Definition. A set X is called <u>countable</u> if $X \simeq \mathbf{N}$.

Exercise:

1. Prove that the set of even integers is countable.

2. Prove that the set of rational numbers with denominator 2 is countable.

3. Prove that the set of rational numbers with denominator 3 is countable.

Observation: If the elements of a set X can be listed in order, say $X = \{x_0, x_1, x_2, x_3, \ldots\}$ so that

i. every element of X is represented is the ordering
ii. no element is repeated
iii. X is infinite

Then $f(x_i) \to i$ is a function since by i) each element of X is in the ordering. Now, f bijection since by ii) f is one to one and by iii) f is onto $\mathbf{N}$.

Exercise 2: Recall the above observation is how we explained that $\mathbf{Z}$ is countable. That is we listed $\mathbf{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$.

Exercise 3: a. Describe an ordering of the $\mathbf{N} \times \mathbf{N}$.

$(0, 0)$ $(0, 1)$ $(0, 2)$ $(0, 3)\ldots$
$(1, 0)$ $(1, 1)$ $(1, 2)$ $(1, 3)\ldots$
$(2, 0)$ $(2, 1)$ $(2, 2)$ $(2, 3)\ldots$
$(3, 0)$ $(3, 1)$ $(3, 2)$ $(3, 3)\ldots$
: : :

True or False: $\mathbf{N} \times \mathbf{N} \simeq \mathbf{N}$

b. Let f: N-$\{0\} \times \mathbf{N}$-$\{0\} \to \mathbf{N} - \{0\}$ be defined as $f((a,b)) = 2^{a-1}(2b-1)$.

Evaluate $f((1,1))$ = _____
Evaluate $f((2,1))$ = _____
Evaluate $f((1,2))$ = _____
Evaluate $f((3,1))$ = _____
Evaluate $f((1,3))$ = _____
Evaluate $f((2,2))$ = _____

Prove that f is a bijection.

Theorem.

(a) Any subset of a countable set is finite or countable.

(b) Any infinite set has a countable subset

(c) The union of a finite or countable family of finite or countable sets is finite or countable.

Proof. We read and discussed proof based on textbook proof.

Exercise 4: Prove that the set of rational numbers is countable.

Notation. X^2 denotes $X \times X$, X^3 denotes $X \times X \times X$, and so on.

i. True or False: $\boldsymbol{N}^2 \simeq \boldsymbol{N}$

ii. True or False: $\boldsymbol{N} \simeq \{0\} \times \boldsymbol{N}^2$

iii. True or False: $\boldsymbol{N} \simeq \{1\} \times \boldsymbol{N}^2$

iv. True or False: $\boldsymbol{N}^3 = \{\{i\} \times \boldsymbol{N}^2 : i \in \boldsymbol{N}\} = \bigcup_{i \in \boldsymbol{N}} \{i\} \times \boldsymbol{N}^2$

v. True or False: $\boldsymbol{N}^3 \simeq \boldsymbol{N}$

Theorem: Let n be a positive integer greater than 1. Prove $\boldsymbol{N}^n \simeq \boldsymbol{N}$.

Proof. Exercise 3 a. provides our basis step that $\boldsymbol{N}^2 \simeq \boldsymbol{N}$. Assume that $\boldsymbol{N}^k \simeq \boldsymbol{N}$ for k some positive integer greater than 2. Since $\boldsymbol{N}^{k+1} = \{ \{i\} \times \boldsymbol{N} \times \ldots \times \boldsymbol{N} : i \in \boldsymbol{N} \}$ and each $\{i\} \times \boldsymbol{N} \times \ldots \times \boldsymbol{N}$ is countable by inductive assumption, $\boldsymbol{N}^{k+1}$ is the countable union of countable sets. Thus, $\boldsymbol{N}^{k+1}$ is countable.

i. True or False: $\mathbf{Q} = \{ x : x \text{ is a solution of } ax + b = 0 \text{ for } a \in \boldsymbol{Z} - \{0\} \text{ and } b \in \boldsymbol{Z}\}$

ii. True or False: $\{ x : x \text{ is a solution of } ax + b = 0 \text{ for } a \in \boldsymbol{Z} - \{0\} \text{ and } b \in \boldsymbol{Z}\} \simeq \boldsymbol{N}$

Definition: **Z**[x] denotes the set of polynomials with integer coefficients. A real number x is said to be algebraic if x is a root of a nonzero polynomial with integer coefficients.

Homework

1. Find a bijection f: $N \times \mathbf{N} \rightarrow \mathbf{N}$.
2. Prove that $\mathbf{Z}^k$ is countable.
3. Prove that $\mathbf{Q}^k$ is countable.
4. Argue that the set of all finite sequences of integers is countable.
5. Argue that the set of all algebraic numbers is countable.

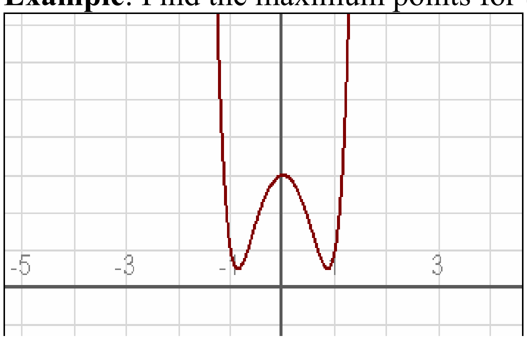
i. True or False: The disjoint intervals , …, (1, 1.5), (1.5, 2), (2, 2.5), (2.5, 3), … each have at least one rational number in the interval.

ii. True or False: The disjoint intervals , …, (1, 1.5), (1.5, 2), (2, 2.5), (2.5, 3), … are countable.

Exercise. Prove that for any collection of disjoint intervals *(p, q)* where *p* and *q* are reals numbers is finite or countable.

Definition. A point $x \in$ **R** is called a maximum point for a function f: **R** $\rightarrow$ **R** if there exists some $\varepsilon > 0$ such that $f(x) > f(x + h)$ for any h such that $|h| < \varepsilon$ and $h \neq 0$.

Example: Find the maximum points for the function in the graph below.



iii. True or False: If x_1 is a maximum points of f, then there exist ε_{x1} such that for any x in $(x_1-\varepsilon_{x1}, x_1+\varepsilon_{x1})$ if $x \neq x_1$, then $f(x_1) > f(x)$.

iii. True or False: If x_1 and x_2 are maximum points for f, then there exist disjoint open intervals I_1 and I_2 such that for any real number x in I_1 if it isn't x_1 then $f(x_1) > f(x)$ and for any real number x in I_2 if it isn't x_2, then $f(x_2) > f(x)$.

Exercise. Prove that the set of all maximum points (for any function f) is either finite or countable.

iv. True or False: If A in infinite and B is countable, then $A \cap B$ is countable or finite.

v. True or False: If A in infinite and B is countable, then $B \setminus A$ is countable or finite.

Theorem: If A is infinite and B is countable (or finite), the union $A \cup B$ has the same cardinality as A.

Proof. Without of loss of generality we may assume that A and B are disjoint (since the intersection $A \cap B$ can be deleted from B and the remaining set $B' = B \setminus A$ is still countable (or finite) (and can be argued with instead).

Let P be a countable subset of A; let Q be the rest that is $Q = A \setminus P$. We have to prove that $B \cup P \cup Q$ has the same cardinality as $P \cup Q$. Both $B \cup P$ and P are countable. Consider a one-to-one correspondence between them and extend it to the one between $B \cup P \cup Q$ and $P \cup Q$ (which is the identity on Q). □

Recall that $(0, 1) \simeq (1, \infty)$ (what was the one-to-one correspondence?)
Recall that $(0, 1) \simeq (0, b)$ (what was the one-to-one correspondence?)
Recall that $(a, b) \simeq (c, d)$ (what was the one-to-one correspondence?)

Example: Prove that $(0, 1) \simeq [0, 1)$. Since $[0, 1) = (0, 1) \cup \{0\}$, $(0, 1)$ is infinite and $\{0\}$ is finite, the above Theorem implies that $(0, 1) \simeq [0, 1)$

Exercise: Prove that $(0, 1) \simeq [0, 1]$.

Exercise: Prove that $(0, 1) \simeq [1, \infty)$.

Example: Prove that $\mathbf{R} \simeq (-\pi/2, \pi/2)$.

Homework:

1. Prove that the real numbers $\mathbf{R}$ and the irrational numbers have the same cardinality.

2. Prove that the real numbers $\mathbf{R}$ and any open interval of real numbers have the same cardinality.

3. Prove that the real numbers $\mathbf{R}$ and $(1, \infty)$ have the same cardinality.