Set Operations

Definitions. Let *A* and *B* be sets.

1. The **union of sets A and B** is denoted and defined as follows: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$
2. Fact: $|A \cup B| = |A| + |B| - |A \cap B|$

3. The **intersection of sets A and B** is denoted and defined as follows: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$

4. Two sets *A* and *B* are said to be **disjoint** if $A \cap B = \varnothing$.

5. The **difference of two sets A and B**, denoted by *A* - *B*, is defined as follows: $A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$.

6. The **complement of a set A**, denoted by $\overline{A}$, is defined as follows: $\overline{A} = \{x \mid x \notin A\}$.

Set Identities: Table 1 on page 49 of our text consists of set identities, many of which we will prove.

Set Identities	
Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws

There are many proof techniques used to prove set identities (we will omit membership tables.) Two of these methods are illustrated in what follows.

Example 1: Prove DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof:

$\overline{A \cup B} =$	
$\{x \mid x \notin (A \cup B)\} =$	By definition of the complement of a set.
$\{x \mid \neg(x \in (A \cup B))\} =$	Symbolic notation of the negation.
$\{x \mid \neg(x \in A \vee x \in B)\} =$	By definition of the union
$\{x \mid \neg(x \in A) \wedge \neg(x \in B)\} =$	???
$\{x \mid (x \notin A) \wedge (x \notin B)\} =$	From symbolic notation to set notation.
$\{x \mid (x \in \overline{A}) \wedge (x \in \overline{B})\} =$	???
$\overline{A} \cap \overline{B}$	By the definition of the intersection of sets.

Hence, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
End of Proof.

Example 2: Exercise #6 (f) Prove $A \cup U = U$ (Note that this is one of the domination laws)

$A \cup U =$	
$\{x \mid (x \in A) \vee (x \in U)\} =$	By definition of the intersection of sets.
$\{x \mid (x \in A) \vee T\} =$	Since the empty set has no elements.
$\{x \mid T\} =$	??
U	

Example 3: Show $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Proof.

$\overline{A \cup (B \cap C)}$	$= \overline{A} \cap \overline{(B \cap C)}$	By DeMorgan's Law for sets.
	$= \overline{A} \cap (\overline{B} \cup \overline{C})$	By DeMorgan's Law for sets.
	$= (\overline{B} \cup \overline{C}) \cap \overline{A}$	??
	$= (\overline{C} \cup \overline{B}) \cap \overline{A}$	

End of Proof.

Example 4: Find the Error in the following proof. Note the statement is true (Exercise #13) but the proof has errors.

Prove that $A - B = A \cap \overline{B}$

Proof.

$A - B$	$= \{x \mid (x \in A) \cap (x \notin B)\}$	By definition of set difference
	$= x \mid (x \in A) \cap (x \in \overline{B})$	By definition of the complement of a set
	$= x \mid A \cap \overline{B}$	By definition of the intersection

End of Proof.

Generalized Unions and Intersections

Let A_1, A_2, …, A_n be sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n.$$

Note: an element is in the union if it is in at least one of the A_i for $i=1, 2, 3, \ldots n$.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n.$$

Note: an element is in the intersection if it is in each of the $_i$ for $i=1, 2, 3, \ldots n$

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Let A 1 2 3 } for $i =$, , ,...

$$\bigcup_{i=1}^{n} A_i =$$

(a) Find $\bigcap_{i=1}^{n} A_i =$

Example 6.

Let $A_i = \{i, i+1, i+2 \ldots\}$ for $i = 1,2,3,\ldots$

(a) Find $\bigcup_{i=2}^{n} A_i =$

(b) Find $\bigcap_{i=2}^{n} A_i =$