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This is a list of conjectures made by Graffiti. As a student and collaborator of Dr. Siemion Fajtlowicz, this is not the first time that I have set the program to run conjectures, however, this is the first time that I will do so independently of Dr. Fajtlowicz (but with his blessing). Please send proofs or counterexamples (they are very important) to delavina@math.uh.edu.

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Throughout this document G will denote a simple, finite graph.

Notation:

$n(G)$ = number of vertices of G.

$\mu(G)$ = the matching number of G

$\alpha(G)$ = the independence number of G

$\delta(G)$ = the minimum degree of G

$\Delta(G)$ = the maximum degree of G

November 1996 Let L denote the maximum number of leaves in any spanning tree of a simple, connected, finite graph. Conjectures 1-8 are lower bounds on L. This concept recently came up in correspondence with Dr. Jerrold Griggs, who posed the following conjecture: $L \geq 1 + n(G) - 2\alpha(G)$. *Proven on 12/96, see proof below in conjecture 7.*

1. $L \geq 1 + n(G) - 2\mu(G)$

This conjecture was proven by **S. Fajtlowicz 11/96**.

Proof. *Let G be a connected graph with the vertex set V. Let M be a maximal matching for which there is a tree spanning the vertices of M, i.e., M is a maximal matching spanning a connected subgraph. Let k be the number of vertices in M. For each vertex v not in M there is vertex u in M such (u, v) is an edge of G. Otherwise, there is a vertex in $V - M$ (if the latter is non-empty) whose shortest distance to a vertex in M is 2, and then M could be extended to a larger matching spanning a connected subgraph.*

For each v in $V - M$ we can add one such edge (v, u) to get a spanning tree with at least $n - k \geq n - 2m$ vertices. If M is not a maximum matching of G then $k < 2m$ and thus $L \geq n - 2m + 2$. If M is maximum then at least one vertex of M is of degree 1 in T, because otherwise we would have an alternating $M-$augmenting path. ∎

2. The **local independence(v)** is the independence number of the subgraph induced by the neighbors of vertex v. L is at least 2(Average of local independence$(v) - 1$).

3. The **temperature of a vertex** v of graph G is $d/(n(G) - d)$, where d is the degree of vertex v, and $n(G)$ is the number of vertices of G. L is at least maximum temperature times the independent domination number.

4. The **span of a set X** is the set of vertices adjacent to at least one vertex of X. Let $S(\overline{e})$ be the cardinality of the span of a nonedge (i.e. a 2-element independent set). $L \geq$ minimum of $S(\overline{e})$ - 1.

> **E.D.** *Proof. We can assume that G is of diameter more than one. Consider two independent vertices of distance 2, and start with two stars emanating from these vertices such that in the resulting binary the two vertices have exactly one neighbor in common. When you extend (in any manner) this binary to a spanning tree you already have a tree with at least minimum of $S(\overline{e})$ minus 1 leaves.* ∎

5. A **sphere S(v,r)** is the set of all vertices whose distance from v is r. $L \geq$ maximum$\{\mid S(v,r) \mid : v$ is a center of $G\}$.

> **S. Fajtlowicz.** *A slightly stronger conjecture is true: you can take the maximum of all spheres, which will be often larger. In general, L(G) = max L(H) where maximum is taken over all connected subgraphs of G. It seems to me that characterization of the equality is of interest in both cases, i.e., in the case of the conjecture and in the case of the fact above.*

> **E. DeLaVina and S. Fajtlowicz** *Let v be a fixed vertex we define $x \leq y$ if and only if there exists a path P(v,y) of length d(v,y) (i.e. shortest distance) from v to y that contains vertex x. The relation clearly defines a partially ordered set. Let v and k be fixed, we define the* **kth surface of a vertex v** *as the set of maximal elements of the poset defined by v and k. Let the* **surface number** *of a graph be the maximum $\{\mid$ kth surface$(v) \mid : v$ a vertex of G and $k \leq$ diameter $\}$. $L \geq$ the surface number $\geq$ maximum$\{\mid S(v,r) \mid : v$ is a vertex of $G\}$.* ∎

6. $L \geq 1 + n - \mu(G) - \alpha(G)$

Proof. This conjecture follows from conjecture 1 and $L \geq n - 2\alpha + 1$*, see proof in conjecture 7. 12/96*

7. $L \geq$ max local independence$(G) - 1 + n - 2\alpha(G)$

E. DeLaVina and S. Fajtlowicz *Proof of Jerry Griggs conjecture. Let d denote the connected domination number of G. It is easily seen that L = n - d. In [FW], Fajtlowicz and Waller have shown that for every graph there exists a connected dominating set D that contains a maximal independent set I such that* $\mid D \mid \leq 2 \mid I \mid -1$*. This implies that* $d \leq 2\alpha - 1$*, and thus* $L \geq n - 2\alpha + 1$*. If the maximum local independence of G is at most two then the statement of conjecture seven easily follows from* $L \geq 2\alpha - 1$*. Thus if conjecture seven is false then the maximum local independence of G is at least three and* $\alpha < n/2$*.*

From the above proof it is now clear that all conjectures of the form $L \geq n - b$*, where b is some invariant, are conjectures of the form that* $d \leq b$*, where d is the connected domination number.12/96*

[FW] Fajtlowicz, S. and W. A. Waller, On Two Conjectures of Graffiti, Congressus Numerantium 55(1986), pp.51-56.

8. **Even(v)** is the number of vertices at even distance from vertex v. $L \geq$ maximum of Even $-\alpha$

Proof. Let c(O(v)) denote the number of components of the subgraph induced by vertices at odd distance from vertex v. For any vertex v we will construct a spanning tree with at least $Even(v) - c(O(v))$ *leaves. Let v be a vertex of G and let D(i) denote the set of vertices at distance i from vertex v. We construct a tree rooted at vertex v, and extend the tree to span vertices in D(1). For every subsequent D(i) where i is even we extend the tree in any manner to include D(i) such that each vertex of D(i) is a leaf in the resulting tree, and for every subsequent D(i) where i is odd we take a vertex u to represent each component of the subgraph induced by vertices of D(i) and join it to a vertex of D(i-1); extend the tree to include the neighbors of u that are in D(i) as neighbors of u in the tree. The spanning tree clearly has Even(v) minus the number of components of the subgraph induced by the vertices at odd distance from vertex v.*

A similar argument shows we have an odd version of this conjecture. More precisely, let c(O(v)) be the number of components of the subgraph induced by vertices at odd distance from vertex v, and c(E(v)) then number of components of the subgraph induced by vertices at odd distance from vertex v, then we have that $L \geq$ *maximum*$\{$*Even(v)-c(O(v)), Odd(v)-c(E(v))*$\}$.

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