Graffiti.pc on the k-independence number of a graph

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Abstract

The k-independence number of a graph is the cardinality of a largest set of vertices that induce a subgraph of maximum degree at most $k-1$. We prove several conjectures made by the computer program Graffiti.pc and present several of the remaining open conjectures.

keywords: k-independence number, independence number, Welsh-Powell, Graffiti.pc, neighbor dominators, induced subgraph.

1 Definitions and Introduction

Given a finite simple graph $G = (V, E)$, an *independent set* is a subset of V such that no pair of vertices in the subset are adjacent. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\beta(G)$. For a positive integer k, a *k-independent set* is a subset I_k of V such that the subgraph induced by I_k has degree at most $k-1$. The cardinality of a maximum k-independent set is called the *k-independence number* of G and is denoted by $\beta_k(G)$. Note, that k-independent sets are

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a generalization of independent sets, and so $\beta_1(G) = \beta(G)$. This generalization was introduced by Fink and Jacobson in 1985. More recently, one finds a survey on this topic in [1]. For $S \subseteq V$, the subgraph induced by S is denoted $G[S]$. For graph theory terms and definitions not explicitly described or defined below, the reader is referred to any current basic graph theory text.

The Graffiti.pc conjecture-making program was written by E. DeLaViña and inspired by a related program called Graffiti which was written by S. Fajtlowicz. Some details of these programs can be found in [3] and [4], and here we simply note that the programs' conjectures take the form of upper and lower bounds for a user selected graph invariant over a user selected graph property.

In recent years, the Graffiti.pc program has been queried for bounds on invariants related to variations on dominating subsets for connected graphs some of these results can be found in [5], [6], [7], [8], [9], [10] and [11]. Related to the topic at hand, in [8] while investigating the k-domination number of a graph, we proved that $\gamma_{a+b} - \gamma_a \leq \beta_b(G)$. However, it was not until late 2011 that Graffiti.pc was queried for conjectures about the 2-independence number for connected graphs. In the paper at hand we present some results on those conjectures. For the full list of Graffiti.pc conjectures and their current status see [12].

We list some known results as we will have occasion to use them.

Theorem 1.1. *(Favaron [13]) For any graph G and any positive integer k,*

$$\gamma_k(G) \leq \beta_k(G).$$

Theorem 1.2. *(Blidia et. al. [2]) For any graph G and any positive integer k,*

$$\beta_k(G) \leq k\beta(G).$$

2 Main Results

Let $d_G(i)$ be the i^{th} term of the degree sequence arranged in nondecreasing order. The *Welsh-Powell invariant* of the complement of G, denoted here

$WP(\overline{G})$, is the largest $k \in 1...n$ such that $k + d_G(k) \leq n$. It is known that $\beta(G) \leq WP(\overline{G}) \leq n - \delta(G)$, see [15]. Graffiti.pc conjectured three statements (#436a, b and c in [12]) that in general suggest that $\beta_2(G) \leq WP(\overline{G}) + 1$ but that under special conditions the plus one can be excluded. In Theorem 2.3 we address one of those special conditions, but first we prove and extend the generalization to β_k.

Theorem 2.1. *Let G be a simple graph and k a positive integer. Then*

$$\beta_k(G) \leq WP(\overline{G}) + k - 1.$$

Proof. Let I be a maximum k-independent set of G. Let v be a vertex in I. Since v has at most $k-1$ neighbors in I, $d_G(v) \leq n - \beta_k(G) + (k-1)$. Since there are $\beta_k(G)$ vertices with the property that $d_G(v) + (\beta_k(G) - k + 1) \leq n$, $WP(\overline{G}) \geq \beta_k(G) - k + 1$. Hence, $\beta_k(G) \leq WP(\overline{G}) + k - 1$. □

To see that the relation in Theorem 2.1 is sharp for each value of k, let G_k be the join of K_{k+1} with the union of two copies of K_k. It is not difficult to see that $n(G_k) = 3k + 1$, $\beta_k = 2k$, and $WP(\overline{G}) = k + 1$.

Before proceeding to our next theorem, note that Conjecture 4 announced in [8] is now a corollary to Theorem 2.1 and Theorem 1.1. So we record this as our next corollary.

Corollary 2.2. *Let G be a simple graph.*

$$\gamma_2(G) \leq WP(\overline{G}) + 1.$$

A vertex v is called a *neighbor dominator* if there exists a vertex u such that $N[u] \subseteq N[v]$; in [14] such vertices were simply called dominators, moreover they demonstrate that for each such vertex there exists a maximum independent set that does not contain it.

Graffiti.pc's conjecture #436a proposed that if a graph has no neighbor dominators, then the bound for β_2 given in Theorem 2.1 can be improved slightly. We prove this next, but observe that for $\beta = \beta_1$, complete bipartite graphs with parts of different orders show that the result cannot be extended to $k = 1$.

Theorem 2.3. *Let G be a simple n-vertex graph with no neighbor dominators. Then*

$$\beta_2(G) \leq WP(\overline{G}).$$

Proof. Let I be a maximum 2-independent set of G. For every $v \in I$ that is isolated in I, $d_G(v) \leq n - \beta_2(G)$. For any $v \in I$ that has a neighbor v' in I, clearly neither has any other neighbors in I. Moreover, if either v or v' had degree $n - \beta_2 + 1$, then one of them would be a neighor dominator of the other. Therefore, every non-isolated vertex in I also has degree at most $n - \beta_2(G)$. Thus, when G has no neighbor dominators, there are at least $\beta_2(G)$ vertices such that $d_G(v) + \beta_2(G) \leq n$ from which is follows that that $WP(\overline{G}) \geq \beta_k(G)$.

□

Our next theorem also originated as a conjecture of the program specifically number 445a in [12]. However, we first note it offers an interesting corollary, namely, if G has a vertex of degree 2 that is adjacent to two vertices of degree 2, then $\beta_2(G) < \beta_3(G)$.

Theorem 2.4. *Let G be a simple graph and let H_2 be the subgraph induced by the vertices of degree at most 2. Then*

$$\beta_2(G) \leq \beta_3(G) - \Delta(H_2) + 1.$$

Proof. If $\Delta(H_2) \leq 1$, the relation easily follows since $\beta_2(G) \leq \beta_3(G) \leq \beta_3(G) - \Delta(H_2) + 1$. So we can assume that $\Delta(H_2) = 2$. Now let v be a vertex of maximum degree in H_2. Then there must exist two other vertices, say a and b, both adjacent to v. Let I be a maximum 2-independent set of G. If $v \notin I$, then since $I \cup \{v\}$ is a 3-independent set, $\beta_2(G) + 1 \leq \beta_3(G)$ from which the relation follows. Now suppose that $v \in I$. Then at least one of a or b is not in I. Without loss of generality suppose $a \notin I$. Then since a is of degree 2, $I \cup \{a\}$ is a 3-independent and again the relation follows. □

Corollary 2.5. *Let G be a simple graph. If G has a vertex of degree 2 that is adjacent to two vertices of degree* 2*, then*

$$\beta_2(G) < \beta_3(G).$$

Recall that a vertex v is called a *neighbor dominator* if there exists a vertex u such that $N[u] \subseteq N[v]$. Let D be the set of neighbor dominators of G. Then $\eta(G) = |D|$ is number of neighbor dominators of G.

It is easily seen that $\beta_k(G) \leq k\beta(G)$, thus Graffiti.pc's next conjecture (#458 in [12]) seems of particular interest as it provides a sufficient condition for equality $\beta_2(G) = 2\beta(G)$.

Theorem 2.6. *Let G be a simple graph. Then*

$$\beta_2(G) \geq 2\beta(G) - \eta',$$

where $\eta' = n - \eta(G)$.

Proof. Let I be a maximum independent set of G and let D be the set of neighbor dominators of G. Next let $A = I \cap D$ (the vertices of I that are neighbor dominators) and $B = I \setminus D$ (the vertices of I that are not neighbor dominators). Let v be a vertex in A. Then there exists a vertex v' in $V \setminus I$, whose closed neighborhood $N[v'] \subseteq N[v]$ and so v' has no neighbors in $I \setminus \{v\}$. Next, let v_i and v_j be in A and suppose that the neighbors that they dominate are v_i' and v_j', respectively. Clearly, $v_i' \neq v_j'$. Moreover, notice that $v_i' \not\sim v_j'$, otherwise $v_j' \in N[v_i'] \subseteq N[v_i]$ implies $v_j' \sim v_i$ which contradicts that $N[v_j'] \subseteq N[v_j]$.

For each $v \in A$ pick one dominated neighbor $v' \in V \setminus I$ to put into set A'. By construction we have $|A'| = |A| = |I \setminus B|$; we have already argued that A' is an independent set and that $I \cup A'$ is a 2-independent set. Now since $\eta' \geq |B|$, we see that $\beta_2(G) \geq |I \cup A'| = |I| + |A'| = |I| + |I \setminus B| = 2|I| - |B| \geq 2\beta(G) - \eta'$ which completes our proof.

□

Corollary 2.7. *Let G be a simple graph. If every vertex of G is a neighbor dominator, then*

$$\beta_2(G) = 2\beta(G).$$

Proof. Assume G is a graph in which every vertex is a neighbor dominator, then $\eta' = 0$, which from Theorem 2.6 yields $\beta_2(G) \geq 2\beta(G)$. Since $\beta_2(G) \leq 2\beta(G)$, the result follows. □

To see that the bound in Theorem 2.6 is sharp, let H_{2m} be the graph constructed by starting with a path on $2m$ vertices labeled left to right $1 \sim 2 \sim ... \sim 2m$. Then we identify a vertex of a copy of K_3 to each vertex of the path with an even label. The number of vertices of H_{2m} is $4m$, and it is not difficulty to see that $\beta(H_{2m}) = 2m$, $\eta'(G) = m$ (the number of vertices that are not neighbor dominators), and $\beta_2(H_{2m}) = 3m$.

To see that a graph G with the property that every vertex is a neighbor dominator is not a necessary condition for $\beta_2(G) = 2\beta(G)$, consider the graph constructed by starting with a path on $4k+2$ vertices and label the edges left to right so that 0 incident to 1, 1 incident to 2 and so on until $4k-1$ incident to $4k$. Then for each edge of the path labeled i where $i \equiv 0 mod 4$ identify an edge of a copy of K_4. The resulting graphs has $6k+2$ vertices, $\beta = 2k+1$, and $\beta_2 = 4k+2$, but not all vertices are neighbor dominators.

Our next result was inspired by a set of conjectures of Graffiti.pc which is reported as conjecture #453 in [12].

Proposition 2.8. *Let G be a simple connected graph, $A \subseteq V$ and $G[A]$ the subgraph induced by A. Then*

$$\beta_2(G) \geq 2c(G[A]) - isol(G[A]),$$

where $c(G[A])$ is the number of components of $G[A]$, and $isol(G[A])$ is the number of trivial components of $G[A]$.

Proof. This is true since one may build a 2-independent set by taking two vertices from each non-trivial component of $G([A])$. □

Graffiti.pc's #454 provided an instance of when the number of trivial components did not detract from the bound in the result above.

Theorem 2.9. *Let G be a simple graph. Then*

$$\beta_2(G) \geq 2c(G[D]),$$

where $c(G[D])$ is the number of components of the subgraph induced by the set of neighbor dominators of G.

Proof. Let I be the set constructed by including a representative vertex v_i from each component of $G[D]$ and also one dominated neighbor, call it w_i, for each v_i. Since the v_is are in different components, they induce an independent set. Now, no w_i is adjacent to any v_j (with $j \neq i$) since $N(w_i) \subseteq N(v_i)$ but v_j is not adjacent to v_i. Lastly, the w_is must also form an independent set otherwise some $w_i \in N(v_j)$ for some $j \neq i$ (since w_j is dominated by v_j), which we just argued was not possible. $\square$

Our next theorem is a generalization of Graffiti.pc's #451 in [12]. Note the program conjectured the relation specifically for 2-independence and the set of minimum degree vertices.

Theorem 2.10. *Let G be a simple connected graph, k a positive integer and let $A \subseteq V$. Then*

$$\beta_k(G) \leq k\beta(G[V \setminus A]) + |A|.$$

Proof. Let D_k be a maximum k-independent set of G, and let $G' = G[V \setminus A]$. Next let $D'_k = D_k \cap (V \setminus A)$. Now by construction and since $\beta_k(G') \leq k\beta(G')$, it follows that $\beta_k(G) = |D_k| = |D'_k| + |D_k \cap A| \leq \beta_k(G') + |A| \leq k\beta(G') + |A| = k\beta(G[V \setminus A] + |A|$. $\square$

To see that the upperbound on β_k in Theorem 2.10 is sharp and sometimes better than the bound $k\beta$, let k be a positive integer and construct G_k by starting with 2 cliques of order $k+2$ and identify them at a vertex. Then join a K_1 to a vertex, but not the identified vertex, of each of the 2 cliques. The number of vertices of G_k is $2k+4$, $\beta_2(G_k) = 2k+1$, and $\beta(G_k) = 3$. Let A be the singleton set containing the minimum degree vertex of G_k. Then $\beta(G_k[V \setminus A]) = 2$, and $k\beta(G_k[V \setminus A]) + |A| = 2k+1 < k\beta(G_k) = 3k$.

Our last result is an example of a false conjecture.

Conjecture 2.11. (#445b) *Let G be a simple connected graph and $G[H_3]$ the subgraph induced by the set of vertices of degree at least 3. Then*

$$\beta_2(G) \leq \beta_3(G) - \lfloor \frac{c(G[H_3])}{3} \rfloor,$$

where $c(G[H_3])$ is the number of components of $G[H_3]$.

The above listed conjecture is false for the graph constructed by taking the union of three copies of $K_{4,4}$ linked together by one edge between the first and second copy and one edge between the second and third and then subdividing the linking edges. It is easily seen that $\beta_2 = \beta_3 = 14$ and that $c(G[H_3]) = 3$.

3 Open Conjectures

Graffiti.pc made several conjectures (see #459 a-f in [12]) that inspired the following more general conjecture.

Conjecture 3.1. *Let G be a simple connected graph, and let $A \subseteq V$. Then*

$$\beta_2(G) \geq |A| - \frac{|E(G[A])|}{2},$$

where $E(G[A])$ is the set of edges of the subgraph induced by A.

Conjecture 3.2. (#449) *Let G be a simple connected graph and H_3 the set of vertices of degree at least 3. Then*

$$\beta_2(G) \leq |V \setminus H_3| + \lfloor \frac{|E(G[H_3])| - 1}{2} \rfloor.$$

$E(G[H_3])$ is the set of edges of the subgraph induced by the vertices of degree at least 3.

Conjecture 3.3. (#452) *Let G be a simple connected graph and let S the set of support vertices of G. Then*

$$\beta_2(G) \leq 2\beta(G) - \mu(G[S]).$$

$\mu(G[S])$ is the matching number of the subgraph induced by the support vertices and $\beta(G)$ is the independence number of the graph G.

Let $d_1 \leq d_2 \leq ... \leq d_n$ be the degree sequence of graph G arranged in non-decreasing order. The *annihilation number*, denoted A, is the largest integer k such that the sum of the first k terms of the sequence is at most half the number of edges. The *residue* R of a graph G of degree sequence $d_1 \geq d_2 \geq ... \geq d_n$ is the number of zeros obtained by the iterative process consisting of deleting the first term d_1 and of subtracting 1 from the next d_1 terms and sorting the new sequence.

Conjecture 3.4. (#447b) *Let G be a simple connected graph, A the annihilation number of G and R the residue of G. Then*

$$\beta_2(G) \leq A + \frac{1}{2}(R+1).$$

Conjecture 3.5. (#441a) *Let G be a simple connected n-vertex graph and A the intersection of all maximum independent sets of G (called the core of G). Then*

$$\beta_2(G) \leq n - \mu(G[N(A)]) - 1,$$

where $\mu(G[N(A)])$ is the matching number of the subgraph induced by the neighbors of the core of G.

Conjecture 3.6. (#441b) *Let G be a simple connected n-vertex graph and A the intersection of all maximum independent sets of G (called the core of G). Then*

$$\beta_2(G) \leq n - \delta(G[N(A)]) - 1,$$

where $\delta(G[N(A)])$ is the minimum degree of the subgraph induced by the neighbors of the core of G.

Conjecture 3.7. (#440) *Let G be a simple connected graph and let A the set of minimum degree vertices of G. Then*

$$\beta_2(G) \leq n - \gamma(G[V \setminus A]),$$

where $\gamma(G[V \setminus A])$ is the domination number of the subgraph induced by the non-minimum degree vertices of G.

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