Introductory Graph Theory

I. Basic Concepts

1. Definition of a Graph

Intuitive Definition: A ***simple graph*** is a collection of vertices (visualized as dots) and edges (visualized as arcs between dots).

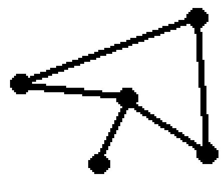


Figure 1.1

Formal Definition: A ***simple graph*** G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$, where $E(G) \subseteq V(G) \times V(G)$, that is each edge is an unordered pair of vertices.

Definitions: A ***vertex v is incident to an edge*** if v is one of the pair of vertices which determines the edge. The ***degree of a vertex*** is the number of edges to which it is incident. We denote the degree of a vertex v as **deg(v)**.

Given a graph G on vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, the ***degree sequence of G*** is **deg(v_1)**, **deg(v_2)**, ..., **deg(v_n)**. The maximum value of the degree sequence is the ***maximum degree of the graph*** and we denote it by $\Delta(G)$. The minimum value of the degree sequence is the ***minimum degree of the graph*** and we denote it by denoted by $\delta(G)$.

Example : Suppose G is a graph on vertex set $\{a, b, c, d, e\}$ and edge set $\{(a, b), (b, c), (c, d), (d, e), (d, a)\}$. Note that the maximum degree of the graph is 3 and the minimum degree is 1. This graph can be visualized using figure 1.1 by labeling the vertices and edges as follows:

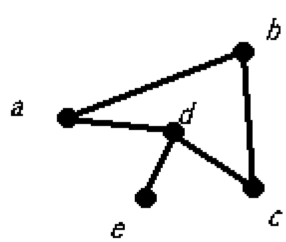
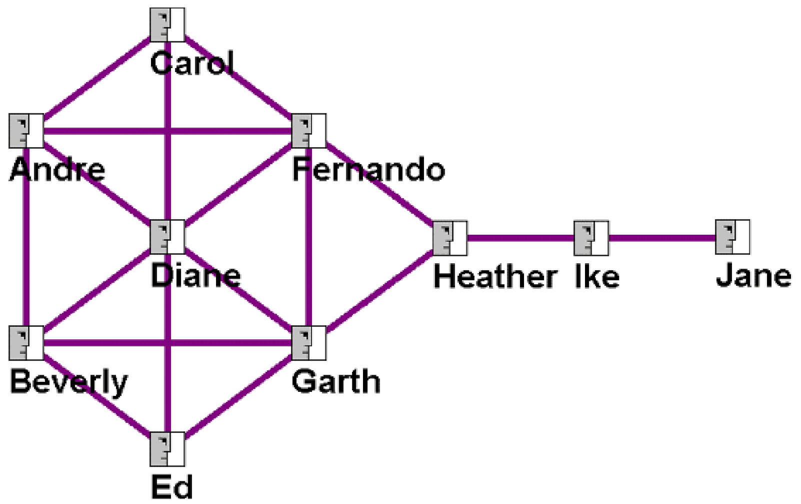


Figure 1.2

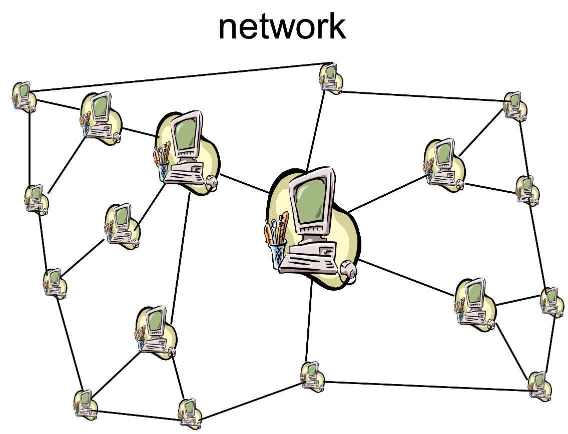
2. Applications of Graphs

Example. Acquainance Graph. Suppose that the people are vertices and that there is an edge between two people if they are acquainaces.



Determine each of the following for the graph above.
The degree sequence____ ,
The number of vertices _____ ,
The maximum degree ______,
The minimum degree ______,

Example



Determine each of the following for the graph above.
The degree sequence____ ,
The number of vertices _____ ,
The maximum degree ______,
The minimum degree ______,

3. Some Special Graphs and More Graph Concepts and Invariants

A *graph concept* is a property of the graph. A *graph invariant* is a numeric value associated with the graph, usually independent of the way the graph is drawn.
Definitions: Let G be a graph on vertex set V and edge set E. For any two vertices, say u and v, if (u, v) is an edge we say that ***u* is adjacent to *v***. To indicate that two vertices u and v are adjacent we use the notation $u \sim v$.

The ***empty graph on n vertices*** (also called the null graph on n vertices) is the graph on n vertices with no edges.

Example. Let G be the empty graph on 5 vertices.
The degree sequence_______________________________________ ,
The number of vertices _____ ,
The maximum degree ______,
The minimum degree ______,

The ***complete graph on n vertices*** is the graph on n vertices in which every two vertices are adjacent. We use the notation $\boldsymbol{K_n}$ to denote the complete graph on n vertices.
Example. Let G be the complete graph on 6 vertices.
The degree sequence_______________________________________ ,
The number of vertices _____ ,
The $\Delta(G)$ = ______,
The $\delta(G)$ = ______,

Perhaps you wondered why we introduced no notation for the empty graph? The following definition provides one motivation.

Definition: Let G be a graph on vertex set V and edge set E. We define the ***complement graph of G*,** denoted $\overline{G}$, as a graph on the the same vertex set V in which two vertices adjacent in $\overline{G}$ if and only if they are not adjacent in G.

With some thought it is easily seen that the empty graph on n vertices is denoted by $\overline{K_n}$.

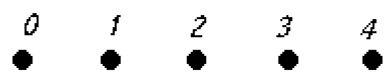
Exercise : Draw $\overline{G}$ if G is the graph drawn below.

Exercise : Find a graph G (on four vertices) for which G and $\overline{G}$ can be drawn so they *appear* the same; formally this should be so that $G \simeq \overline{G}$. The symbol $\simeq$ is used to indicate that the two graphs are essentially the same except for the way the vertices are labeled (formally the symbol $\simeq$ is read as *isomorphic to*, which we will investigate soon.)

Exercise : Suppose we are given a graph on vertex set $\{0,\ 1,\ 2,\ 3, 4\}$ and that edges of this graph are determined by the following rule:

$$\forall x, y \in \{0,1,2,3,4\}\ x \sim y \ \leftrightarrow |x-y| = 1$$

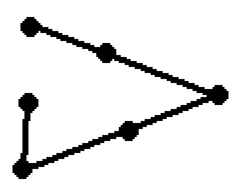
Draw the edges subject to the given rule in the following diagram.



The graph in the above Exercise is an example (with $n = 5$) of a graph called a ***path on n vertices***, denoted P_n.

Definition: A graph is called a ***Path on n vertices*** if the vertices can be labeled with elements of $\{0, 1, 2, ..., n-1\}$ so that the edge set is $\{(i, i+1) | i \in \{0, 1, 2, ..., n-2\}\}$. Such a graph is denoted by P_n.

Exercise 1: Verify that the following graph is P_6. Hint: don't let the shape of the drawing sway your thoughts, use the definition.



Definition: A graph is called a ***cycle on n vertices*** ($n \geq 3$) if the vertices can be labeled with elements of $\{0, 1, 2, ..., n-1\}$ so that the edge set is $\{(i, i+1) | i \in \{0, 1, 2, ..., n-1\}\} \cup \{(0, n-1)\}$. Such a graph is denoted by C_n.

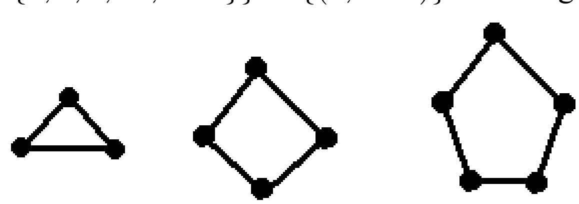
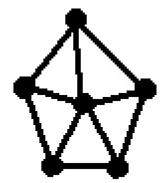


Figure 1.3. $C_3, C_4,$ and C_5

Definition: A graph is called a ***Wheel on n vertices*** if the vertices can be labeled with elements of $\{0, 1, 2, \ ..., n-1\}$ so that the vertices $\{1, 2, ..., n-1\}$ determine a cycle on $n-1$ vertices and vertex 0 is adjacent to each of the vertices in $\{1, 2, ..., n-1\}$ Such a graph is denoted by W_n.

Exercise : Verify that the following graph is W_6.



Definition. The ***intersection graph*** of a collection of sets A_1, A_2, ..., A_n is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing the two sets if these sets have a nonempty interestion.

Definition. The ***n-Cube graph,*** denoted Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if an donly if the bit strings that they represent differ in exactly one position.