

Sequences and Summations

Def. Recall the natural numbers are denoted $N = \{1, 2, 3, \dots\}$ and the whole numbers $W = \{0, 1, 2, 3, \dots\}$. Let S be a subset of integers. A **sequence** is a function $f: N \rightarrow S$ or $f: W \rightarrow S$. Instead of $f(n)$ we use the notation a_n to denote the image of the integer n . We call a_n the n^{th} term of the sequence.

Sequences and summations occur in many Math and CS areas of study for example Linear algebra, Calculus, Number Theory, Complexity of Algorithms and Discrete Structures.

Notation: We use the notation $\{a_n\}$ to describe the sequence a_1, a_2, a_3, \dots .

Example: Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ (*the general formula of the sequence.*)

The list of the terms of this sequence, beginning with a_1 , namely,

$$a_1, a_2, a_3, a_4, \dots$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

Example: Consider the sequence $\{b_n\}$, where $b_n = (-1)^n$ (*also called the rule of the sequence.*)

The list of the terms of this sequence, beginning with b_1 , namely,

$$b_1, b_2, b_3, b_4, \dots$$

starts with

$$1, -1, 1, -1, 1, \dots$$

Exercise: What are the a_0, a_1, a_2, a_3 terms of the following sequences?

- i. $\{4^n\}$
- ii. $\{\lfloor \frac{n}{2} \rfloor\}$
- iii. $\{n!\}$

Def. Finite sequences $a_1, a_2, a_3, \dots, a_n$ are called **strings**. A string is also denoted by $a_1 a_2 a_3 \dots a_n$ (without the commas.)

Special Integer Sequences

Arithmetic sequences are those such that consecutive differences are constant. The general formula for the n th term is $a + d(n-1)$ where a is the first term and d is the constant difference.

Example: 1, 3, 5, 7, 9, ...

Find the n^{th} term (i.e. the general rule) _____

The 100^{th} term is _____

Example: 0, 50, 100, 150, 200, ...

Find the n^{th} term _____

The 100^{th} term is _____

Geometric sequences: are those such that consecutive ratios (2^{nd} term divided by the 1^{st} , 3^{rd} term divided by the 2^{nd} , etc.) are constant. The general formula for the n th term is ar^{n-1} where a is the first term and r is the constant ratio.

Example: 3, 6, 12, 24, 48, ...

Find the n^{th} term _____

Find the 100^{th} term _____

Example: 10, 100, 1000, 10,000, 100,000, ...

Find the n^{th} term _____

The 100^{th} term is _____

Suppose that we are given a numerical sequence that is neither an arithmetic nor geometric sequence. What to do then?

Other tool to consider for discovering patterns:

- Are the terms dependant on previous terms?
- Are the terms squares? Or cubes?
- Are some terms repeated?

Example: 1, 8, 27, 64, 125, ...

Find a simple formula (or general rule) for the sequence.

Example: 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, ...

Find a simple formula (or general rule) for the sequence.

Example: 1, 1, 2, 3, 5, 8, 13 ... (**Fibonacci sequence**)

Find a simple formula (or general rule) for the sequence.

Summations

Notation: The sum $a_1 + a_2 + a_3 + \dots + a_n$ is represented by the

summation notation $\sum_1^n a_i$. The variable i is called the **index of summation**.

$\sum_{i=1}^n a_i$ is read as "the sum for i equal 1 to n ."

$\sum_{j=m}^n a_j$ is read as "the sum for j equal m to n ."

n is called the **upper limit**.

m is called the **lower limit**.

The Greek letter sigma is used to denote summation

$$\sum_m^n a_j$$

Exercise: Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$, for $n = 1, 2, 3, \dots$

Exercise: What is the value of $\sum_{j=5}^7 j^2$?

Exercise: What is the value of $\sum_{j=0}^4 2^j$?

Exercise: What is the value of $\sum_{k=1}^5 2^{(k-1)}$?

Manipulating the Notation:

- Suppose we have the sum $\sum_{j=1}^7 j^2$ and that we need to change

the summation notation so that the lower limit is 0. Note that this is accomplished by letting $m = j-1$. Then $j = m+1$, and

$$\sum_{j=1}^7 j^2 = \sum_{m=0}^6 (m+1)^2 .$$

- By associativity, $\sum_{j=1}^6 j^2 + 7^2 = \sum_{j=1}^7 j^2 = 1 + \sum_{j=2}^7 j^2$

- Note: $\sum_{j=1}^6 j^2 + 7^2 \neq \sum_{j=1}^6 (j^2 + 7^2)$ (Beware of parenthesis)

- By the distributive property of multiplication over addition,

$$3 \left(\sum_{k=1}^n k \right) = \sum_{k=1}^n 3k$$

Example: The sum of the first n terms of a geometric sequence is called a **geometric progression**.

Fact: $\sum_{j=0}^n ar^j = \frac{ar^{n+1} - a}{r-1}$ if $r \neq 1$ (See text p. 74)

Exercise: Find the sum $\sum_{j=1}^{99} \frac{1}{\sqrt{j} + \sqrt{j+1}}$

Double Summations

$$\sum_{i=1}^3 \sum_{j=0}^2 ij =$$

A set index summation

$$\sum_{s \in \{0,1,3\}} s^2 =$$