Exploring Graph Theory Through Conjectures of Graffiti

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Project Description and Background:

Start date: February 2, 2001

Prior to beginning this project, my only experience with Graph Theory was two lectures within a larger Discrete Math class. My senior project advisor, Dr. Ermelinda DeLaVina, suggested an exploration of graph theory through conjectures from the program Graffiti (see description below). We began with a base graph of K1, which is simply a single vertex, and requested conjectures in relation to the fixed concepts of independence (α_G) and clique (ω_G). My task is to prove or disprove the conjectures presented. My counterexamples are added to the database, and a new list of conjectures is generated.

Program Description:

The computer program Graffiti is a creation of Dr. Siemion Fajtlowicz of the University of Houston that was developed jointly with Dr. DeLaVina. This program takes as input a database which can be thought of as a two dimensional array indexed by models (in this case graphs) and concepts (in this case graph concepts). With this information, the program returns graph theoretical conjectures

For this project, we have fixed the concept of the sum of the **independence number (α_G)** of the graph, G, and the **clique number (ω_G)** of G. All conjectures will be of the form:

Let G be some connected graph.

$$a_G + w_G \geq \text{Some algebraic expression composed of other graph concepts}$$

Current Data:

Total number of conjectures presented: 45

Conjectures proven: 6

Conjectures disproven: 25

(Counterexamples used: 20)

Open conjectures:□ 11□

Unresolved conjectures removed by program: 3

Definitions of non-standard terms used:

Minimode: least commonly occurring value in sequence

Length:

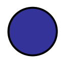
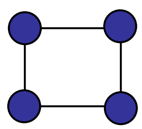
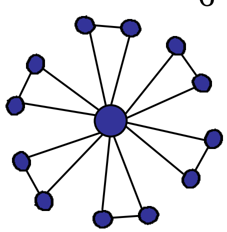
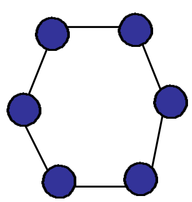
$$\text{Length} = \sqrt{\left(\sum_{v \in V_G} \deg(v)^2\right)}$$

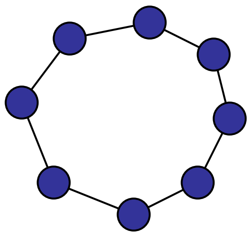
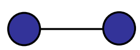
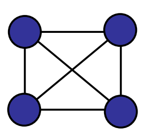
Average distance: For $u,v \in V_G$, $d(u,v)$ is the shortest u,v-path. Average distance is the average of the $d(u,v)$, " $u,v \in V_G$, $u^1 v$

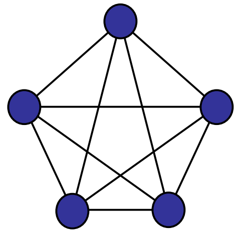
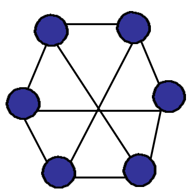
Jump of degree: Put the degree sequence in ascending order, then a jump is the difference between consecutive values.

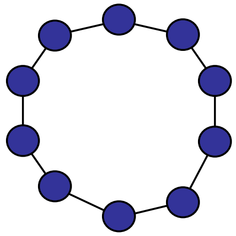
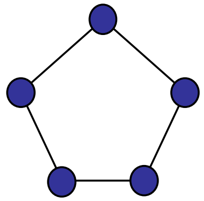
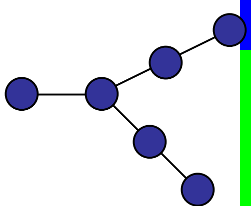
Harmonic mean: Let k be the number of non-isolated vertices of G. Then,

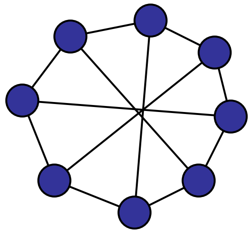
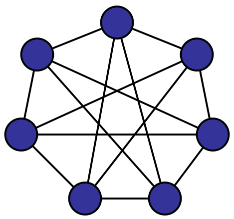
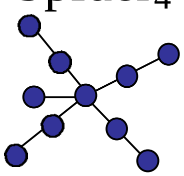
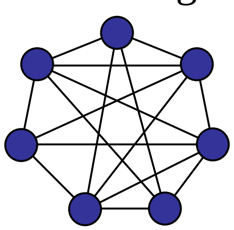
$$\text{Harmonic mean} = \frac{k}{\sum_{v \in V_G} \deg(v)^{-1}}$$

Session	Models	Conjectures	Status
1	K_1 	$a_G + w_G \geq 1 +$ number of vertices of G	*Counterexample:* C_4
2	C_4 	$a_G + w_G \geq$ Maximum degree - minimum degree (range of degree sequence)	*Counterexample:* *Windmill*$_6$
		$a_G + w_G \geq$ number of vertices of G	*Counterexample:* C_6
3	Windmill$_6$  C_6 	$a_G + w_G \geq$ (frequency of minimode of degree sequence of G) - 1	*Counterexample:* C_8
		$a_G + w_G \geq 1 + \left(\sum_{v \in V_G} \deg(v) \right)^{-1}$	**Proved**
		$a_G + w_G \geq 1 +$ number of distinct values of degree sequence of G	See session 6

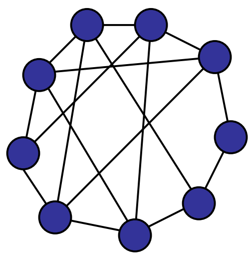
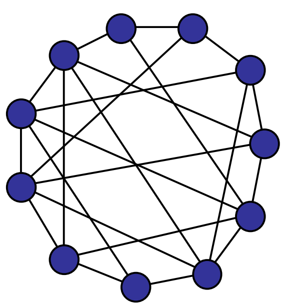
		$a_G + w_G \ge \left(\sum_{v \in V_{\overline{G}}} \deg(v) \right)^{-1}$	**Proved**
4	C_8 	$a_G + w_G \ge 2 + \sum_{v \in V_G} \deg(v)^{-1}$	*Counterexample: K_2*
		$a_G + w_G \ge \frac{1}{2} \left(\sum_{v \in V_G} \deg(v) \right)$	*Counterexample: K_4*
5	K_2  K_4 	$a_G + w_G \ge \frac{1}{2}$(number of vertices of G) + mode of degree sequence of G	*Counterexample: K_5*
		$a_G + w_G \ge 1 +$ number of isolated vertices of $\overline{G}$	**Proved**
		$a_G + w_G \ge 1 +$ minimum degree of $\overline{G}$	Unresolved, but removed by program
		$a_G + w_G \ge 1 + \sum_{v \in V_G} \deg(v)^{-1}$	See session 7

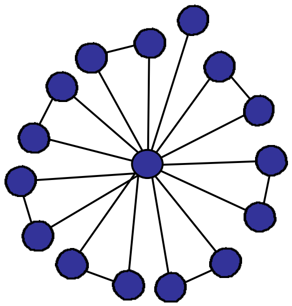
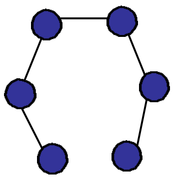
6	K_5 	$a_G + w_G \geq \frac{1}{2}$(number of non-isolated vertices of $\overline{G}$) + mode of degree sequence of G	*Counterexample: Pie_6*
		$a_G + w_G \geq \frac{3}{2} a_G$	*Counterexample: C_{10}*
		$a_G + w_G \geq 1 +$ number of isolated vertices of $\overline{G}$	**Previously Proven**
		$a_G + w_G \geq 1 + \sum_{v \in V_G} \deg(v)^{-1}$	See session 7
		$a_G + w_G \geq 1 +$ number of distinct values of degree sequence of G	Unresolved, but removed by program
7	Pie_6 	$a_G + w_G \geq \frac{1}{2}$(number of non-isolated vertices of $\overline{G}$) $+ a_{\overline{G}}$	*Counterexample: C_5*
		$a_G + w_G \geq 2 +$ mode of degree sequence of G	See session 8
		$a_G + w_G \geq 1 + \sum_{v \in V_G} \deg(v)^{-1}$	*Counterexample: $Spider_2$*

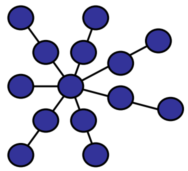
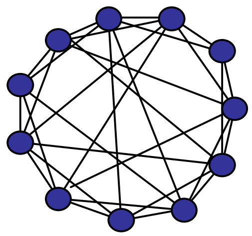
	C_{10} 		
8	C_5 	$a_G + w_G \geq \text{mode of degree sequence} + \left\lfloor \sum_{v\in V_G} \deg(v)^{-1} \right\rfloor$	See session 9
	Spider_2	$a_G + w_G \geq a_G + \ln(\text{length of degrees of G})$	*Counterexample: Ramsey(3,4)*
		$a_G + w_G \geq 1 + \left\lfloor \sum_{v\in V_G} \deg(v)^{-1} \right\rfloor$	See session 9
		$a_G + w_G \geq 2 + \text{mode of degree sequence of G}$	*Counterexample: $CircleStar_7$*
		$a_G + w_G \geq 1 + a_G$	**Proved**
		$a_G + w_G \geq \frac{1}{2}\left(\sum_{v\in V_G} \deg(v)\right)$	Unresolved, but removed by program

Session	Models	Conjectures	Status
9	Ramsey(3,4)  CircleStar$_7$ 	$a_G + w_G \geq \text{mode of degree sequence} + \left\lfloor \sum_{v \in V_G} \deg(v)^{-1} \right\rfloor$	*Counterexample: Spider4*
		$a_G + w_G \geq 1 + \left\lfloor \sum_{v \in V_G} \deg(v)^{-1} \right\rfloor$	*Counterexample: Spider4*
		$a_G + w_G \geq 2 + \text{minimum degree of G}$	*Counterexample: CircleStar$_7$ with 2 edges*
10	Spider$_4$  CircleStar$_7$ with 2 edges 	$a_G + w_G = a_G + a_{\overline{G}}$	**Proved**

11		$a_G + w_G \geq$ mode of degree sequence of $G + a_G - 1$	*Counterexample: Partial CircleStar$_9$*
		$a_G + w_G \geq \frac{1}{2}$(mode of degree sequence of G) $+ a_G$	*Counterexample: G11 with 12 diagonals*
		$a_G + w_G \geq$ Maximum degree - minimum degree (range of degree sequence) - 2	*Counterexample: Windmill$_7$ with 1 leg*
		$a_G + w_G \geq 1 + \frac{1}{2}$(number of vertices of G)	See session 12
		$a_G + w_G \geq$ Radius of $G + 2$	Open
		$a_G + w_G \geq \left\lfloor \sum_{v \in V_G} \deg(v)^{-1} \right\rfloor$	See session 12
		$a_G + w_G \geq 1 +$ number of isolated vertices of $\overline{G}$	**Previously Proven**
		$a_G + w_G \geq 1 + a_G$	**Previously Proven**
		$a_G + w_G \geq$ Diameter of $G + 1$	*Counterexample: Path$_6$*

		$a_G + w_G \geq 1 +$ number of vertices of degree 1 in G	**Proved**
		$a_G + w_G \geq 1 +$ mode of degree sequence of G	See session 12
12	Partial CircleStar$_9$ 	$a_G + w_G \geq 1 + \ln($length of degrees in $\overline{G}) +$ average distance over all vertices of $\overline{G}$	Open
		$a_G + w_G \geq \ln\left(\sum_{v \in V_{\overline{G}}} \deg(v)\right) +$ average distance over all vertices of $\overline{G}$	Open
	G11 with 12 diagonals 	$a_G + w_G \geq \ln\left(\sum_{v \in V_G} \deg(v)\right) +$ average distance over all vertices of G	Open
		$a_G + w_G \geq$ Maximum jump of degree sequence of G - 2	Open
		$a_G + w_G \geq 2($number of distinct values in degree sequence of G$) - 1$	Open

	Windmill$_7$ with 1 leg	$a_G + w_G \geq \frac{1}{2}\left\lfloor \text{length of degrees in G} \right\rfloor$	*Counterexample: Monster*$_{11}$
		$a_G + w_G \geq 2\ln(\text{harmonic mean of } \overline{\text{G}})$	Open
		$a_G + w_G \geq 1 + \frac{1}{4}\left(\sum_{v \in V_G} \deg(v)\right)$	*Counterexample: Monster*$_{11}$
		$a_G + w_G \geq 1 + \frac{1}{2}(\text{number of vertices of G})$	*Counterexample: Monster*$_{11}$
	Path$_6$	$a_G + w_G \geq \text{Radius of G} + 2$	Open
		$a_G + w_G \geq \left\lfloor \sum_{v \in V_G} \deg(v)^{-1} \right\rfloor$	*Counterexample: Spider*$_6$
		$a_G + w_G \geq 1 + \text{average eccentricity of G}$	Open
		$a_G + w_G \geq \text{frequency of degree 1 in degree sequence of G}$	Open
		$a_G + w_G \geq 1 + \text{average degree of G}$	Open
		$a_G + w_G \geq 1 + \text{mode of degree sequence of G}$	*Counterexample: Monster*$_{11}$
		$a_G + w_G \geq 1 + \text{minimum degree of } \overline{\text{G}}$	Open

13	Spider$_6$  Monster$_{11}$ 		□