Propositional Equivalences

Def. A compound proposition that is always true, no matter what the truth values of the (simple) propositions that occur in it, is called **tautology**. A compound proposition that is always false, no matter what, is called a **contradiction**. A proposition that is neither a tautology nor a contradiction is called a **contingency**.

Examples: Let p be a proposition. Indicate whether the propositions are: (A) tautologies (B) contradictions or (C) contingencies.

Proposition	type
$p \vee \neg p$	tautology
$p \wedge \neg p$	contradiction
$p \wedge p$	contingencies

$p \vee \neg p$
$T \vee F$
$F \vee T$

p	$\vee$	$\neg p$
T	T	F
F	T	T

Definition. The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent. Some text books use the notation $p \Leftrightarrow q$ to denote that p and q are logically equivalent.

$p \equiv q$

Objective of the section:

You must learn to determine if two propositions are logically equivalent by the

- truth table method **and**
- by the logical proof method using the tables of logical equivalences (but not true tables)

☆ w/o truth table

Exercise 1: Use truth tables to show that $\neg\neg p \equiv p$ (the double negation law) is valid.

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

$\neg\neg p \equiv p$

Exercise 2: Use truth tables to show that $p \wedge$ T $\equiv p$ (an identity law) is valid.

p	$\wedge$	T
T	T	T
F	F	T

$p \wedge T \equiv p$

T stands for true

Reminiscent

$a + 0 = a$
$a \cdot 1 = a$
$p \vee F \equiv p$

Note: Any equivalence termed a "law" will be proven by truth table, but all others by proof (as we shall see next).

Equivalence	*Name*
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg\neg p \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$(p \to q) \equiv \neg p \vee q$ $\boxed{(p \to q) \equiv \neg q \to \neg p}$	Other useful logical equivalence

Exercise 3: State the name of the law used in the identity

i. $\neg(\neg p \wedge q) \vee T \equiv T$ domination

ii. $T \vee \neg(\neg p \wedge q) \equiv \neg(\neg p \wedge q) \vee T$ commutative

ii. $\neg(\neg p \wedge q) \vee F \equiv \neg(\neg p \wedge q)$ identity

iii. $\neg(\neg p \wedge q) \wedge T \equiv \neg(\neg p \wedge q)$ identity

iv. $\neg(\neg p \wedge q) \equiv \neg\neg p \vee \neg q$ deMorgan

you try

Exercise 4: ***Without*** truth tables to show that

$\neg(\neg p \wedge q) \equiv p \vee \neg q$

proof: $\neg(\neg p \wedge q) \equiv \neg\neg p \vee \neg q$ by deMorgan's
$\equiv p \vee \neg q$ by double Neg □

Exercise 5: ***Without*** truth tables to show that

$[\neg(p \wedge q) \vee (p \wedge q)] \equiv T$

proof: $\neg(p \wedge q) \vee (p \wedge q) \equiv (p \wedge q) \vee \neg(p \wedge q)$ by commut.
$\equiv T$ by Negation law. □

Exercise 6: ***Without*** truth tables to show that

$$\neg p \wedge (p \vee q) \equiv \neg p \wedge q$$

your way

Proof: $\neg p \wedge (p \vee q) \equiv (\neg p \wedge p) \vee (\neg p \wedge q)$ by distrib

$\equiv (p \wedge \neg p) \vee (\neg p \wedge q)$ by commutation

$\equiv F \vee (\neg p \wedge q)$ by Negation

$\equiv (\neg p \wedge q) \vee F$ by commutation

$\equiv \neg p \wedge q$ Identity $\square$

Exercise 7: ***Without*** truth tables to show that

$$\neg (p \vee (\neg p \wedge q)) \equiv \neg (p \vee q)$$

proof: $\neg(p \vee (\neg p \wedge q)) \equiv \neg((p \vee \neg p) \wedge (p \vee q))$ by distrib

$\equiv \neg(T \wedge (p \vee q))$ by Negation law

$\equiv \neg((p \vee q) \wedge T)$ by Commut

$\equiv \neg(p \vee q)$ by Identity $\square$

$p \wedge T \equiv p$

Exercise 8: ***Without*** truth tables to show that $(\neg p \to q) \equiv p \vee q$.

Exercise 9: ***Without*** truth tables to show that $\neg\ (p \rightarrow q) \equiv p \wedge\ \neg q$.

Exercise 10: ***Without*** truth tables to show that $\neg\ (\neg\ p \vee\ (p \vee\ q)) \rightarrow q$ is a tautology.

Exercise 11: ***Without*** truth tables to show that an implication and it's contrapositive are logically equivalent.

Applications

In addition to providing a foundation for theorem proving, which we will cover in this class, this algebraic look at logic can be studied further for the purpose of discussion computer program correctness.