Test 2 2305

Name

Directions: Mark the True and False on this test, however, all other problems should be labeled, rewritten and worked in your blue book.

I. Set Operations
   1. (4 points) Let \( A_i = \{i, i+1, i+2, \ldots, 2i\} \). List the element of the following sets.
      i. \( A_1 = \{1, 2\} \)
      ii. \( A_n = \{n, n+1, \ldots, 2n\} \)
      iii. \( \bigcap_{i=2}^{n} A_i = \emptyset \) since the sets \( A_2, A_3, \ldots, A_n \) have nothing in common
      iv. \( \bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, 2n\} \)

   2. (6 points) Prove the set identity \( A \cup (B \cap C) = (A \cup B) \cap A \) using set identities (provide reasons at each step).

   \[
   \begin{align*}
   A \cup (B \cap C) &= A \cap (B \cap C) \\
   &= A \cap (B \cup C) \\
   &= (A \cup C) \cap A \\
   &= \text{by De Morgan's law} \\
   &= \text{by Commutativity}
   \end{align*}
   \]

II. Functions

1. (4 pts) MATCH the term with its definition in symbolic notation, IF PRESENT.

   \( f : A \rightarrow B \), \( f \) is a function \( \forall y \in B \exists x \in A, f(x) = y \)
   \( f : A \rightarrow B \), \( f \) is a one-to-one function \( \forall x \in A \exists! y \in B, f(x) = y \)
   \( f : A \rightarrow B \), \( f \) is on \( B \) \( \forall x, y \in A, (x \neq y \rightarrow f(x) \neq f(y)) \)

2. (3 pts) Consider the function \( f(n) = \lfloor n/2 \rfloor \) from \( \mathbb{Z} \) to \( \mathbb{Z} \).
   a. What is the domain of \( f \)? \( \mathbb{Z} \)
   b. What is the codomain of \( f \)? \( \mathbb{Z} \)
c. What is the range of $f$?

Even integers, since $\left\lfloor \frac{1}{2} \right\rfloor$ is an integer.

3. (2 points each)

(a) True or False: The relation $f$ from the set of all finite length bit strings to the set of integers, defined by $f(S)$ is the position of the 1 bit in the bit string $S$ is a function.

For example $f(101)$ would be assigned to 1 or 3 which is not a unique assignment.

(b) True or False: The relation $f$ from the set of all finite bit strings to the set of integers, defined by $f(S)$ is the number of 0 bits in the bit string $S$ is a function.

(c) True or False: The function $f: R \rightarrow Z$ defined by $f(x) = \lfloor x \rfloor$ is onto $Z$.

\[ \forall n \in \mathbb{Z} \text{ since } n \in \mathbb{R} \text{ and } \lfloor n \rfloor = n \]

(d) True or False: The function $f: R \rightarrow Z$ defined by $f(x) = \lfloor x \rfloor$ is one-to-one.

\[ f(1.1) = \lfloor 1.1 \rfloor = 2 = \lfloor 2 \rfloor \text{ but } 1.1 \neq 2 \]

4. (5 points) Consider the real valued function $f(x) = 5x - 4$. Prove that this function is one-to-one.

Proof: Let $x, y \in R$. Assume $f(x) = f(y)$. By definition of $f$,

\[ 5x - 4 = 5y - 4 \]

\[ 5x = 5y \]

Since $f(x) = f(y)$, $x = y$. Hence, $f$ is one-to-one.

III. Integers and Division

1. (5 pts)

i. Find GCD($2^4 \cdot 5^4 \cdot 7^3 \cdot 17$, $2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$) = $2^4 \cdot 5^3 \cdot 7^2 \cdot 17$.

ii. Find LCM($2^4 \cdot 5^4 \cdot 7^3 \cdot 17$, $2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$) = $2^4 \cdot 5^4 \cdot 7^3 \cdot 17^3 \cdot 19$.

iii. If the product of two integers $a \cdot b = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3$ and the GCD of $a$ and $b$ is $2^2 \cdot 3 \cdot 5^3 \cdot 7$, then what is the LCM of $a$ and $b$.

Recall that $a \cdot b = \text{gcd}(a, b) \cdot \text{lcm}(a, b)$.

\[ 2^5 \cdot 3^4 \cdot 5^7 \cdot 7^3 = 2^2 \cdot 3 \cdot 5^3 \cdot 7 \cdot \text{lcm}(a, b) \]

\[ \text{lcm} = 2^3 \cdot 3^3 \cdot 5^5 \cdot 7^2 \]
2. (3 pts) Demonstrate the Euclidean Algorithm for finding the gcd of 2024 and 1024. (Note that the gcd is not enough you must demonstrate the Euclidean algorithm).

\[ 2024 = 1 \cdot 1024 + 1000 \]

3. (1 pt each) True/False.

i. True or False: 8 is a divisor of 64.

ii. True or False: 64 is a factor of 8.

iii. True or False: 111 is a prime number.

iv. True or False: For a and b nonzero integers, \( a \mid b \land b \mid c \Rightarrow a \mid c \)

v. True or False: For a nonzero integer \( a, a \mid (b + c) \Rightarrow a \mid c \)

vi. True or False: \( 2 \mid 24 = 12 \)

vii. True or False: \( 13 = 3 \mod 5 \)

viii. True or False: \( 0 = 5 \mod 5 \)

ix. True or False: \( -1 = 19 \mod 5 \)

x. True or False: \( -1 = -16 \mod 4 \)

4. (3 pts) Store the following student ID numbers in memory locations 0-9 using a hashing function \( f(s) = s \mod 10 \). Explain how you managed collisions.

658779, 658772, 657102, 648559, 657991, 657122, 658881, 648557, 657100

\[
\begin{array}{c|c}
\text{loc} & \text{id} \\
\hline
0 & 648559 \rightarrow 658771 \\
1 & 657991 \\
2 & 658772 \rightarrow 657102 \text{ due to collision} \\
3 & 657102 \\
4 & 657112 \\
5 & 658881 \\
6 & 657100 \\
7 & 648557 \\
8 & 658779 \rightarrow 648559 \rightarrow \text{loc 0}
\end{array}
\]

5. (6 pts) Let \( a, b \), and \( c \) be integers such that \( c \) is nonzero. Prove if \( c \mid a \) and \( c \mid b \), then \( c \mid (xa + yb) \) for any integers \( x \) and \( y \).

Proof: Let \( a, b, c \in \mathbb{Z} \) and \( c \neq 0 \). Assume \( c \mid a \) and \( c \mid b \).

By definition of divisibility, \( a = ck \) and \( b = cl \) for some \( k, l \in \mathbb{Z} \). By substitution,

\[
xa + yb = x(ck) + y(cl) = c(xk + yl)
\]

Since \( xk + yl \in \mathbb{Z} \) and \( c \) has closure under multiplication, \( xk + yl \in \mathbb{Z} \). Thus, by definition of divisibility, \( c \mid (xa + yb) \). □
IV. (3 points each) Sequences

i. List the first 6 elements of the sequence \( \left\{ \frac{2^i}{i} \right\}_{i=1} \)

ii. Find a formula for the following sequence -1, 2, 3, 4, 6, 11, 14, 17, ...

\[ a_n = -1 + 3(n-1) \] arithmetic seq

iii. Find a formula for the following sequence 10, 100, 1000, 10000 ...

\[ a_n = 10^n \] geometric seq

V. (3 points each) Summations

iv. Evaluate the double sum \( \sum_{i=0}^{4} \sum_{j=0}^{3} i^2 + j^2 \)

\[ = \sum_{i=0}^{4} \sum_{j=0}^{3} i^2 + j^2 = \sum_{i=0}^{4} (i^2 + j^2) \]

v. Evaluate \( \sum_{i=0}^{1002} i^2 \)

\[ \sum_{i=0}^{1002} i^2 = \frac{1002 \cdot 999}{2} = 497001 \]

vi. Evaluate \( 1^2 + 2^2 + 3^2 + \cdots + 24^2 + 25^2 \)

\[ = \frac{24 \cdot 25 \cdot 49}{6} = 4900 \]

\[ \text{Geometric progression} \]

vii. Write the following sums in summation sigma notation.

\[ a + a^2 + a^3 + a^4 + \cdots + a^9 = \sum_{i=0}^{9} a^i \]

viii. Write the following sums in summation sigma notation.

\[ 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + 9 \cdot 10 = \sum_{i=1}^{9} i(i+1) \]

ix. Express \( \sum_{k=0}^{9} (k+1) \) as a single equivalent summation whose lower limit is 2.

let \( m = k+2 \). Then when \( k=0 \), \( m=2 \) and

when \( k=9 \), \( m=11 \).

Since \( k \geq m-2 \), \( k+1 = m-1 \).

\[ \sum_{m=2}^{11} \]
VI. Mathematical Induction

1. (10 pts) Use the principal of math induction to prove one of the following (i or ii exclusively).

i. \(2 \mid (n^2 + 3n) \quad \forall n \geq 1\)

ii. \(1 + 3 + 9 + 27 + 81 + \ldots + 3^n = \frac{3^{n+1} - 1}{2} \quad \forall n \geq 1\)

i) Let \(P(n) \) denote \(2 \mid (n^2 + 3n)\). Since \(1^2 + 3 \mid 4 \) and \(2 \mid 4\), \(P(1)\) is true. Assume \(P(k)\) is true, that is, assume \(2 \mid (k^2 + 3k)\).

By \(\mathbb{N}\), \((k+1)^2 + 3(k+1) = k^2 + 3k + 2(2k+3)\) \[k^2 + 3k + 2(k+2) = 2(k^2 + 3k + 2) \]

Thus, \(P(k+1)\) is true whenever \(P(k)\) is true. Hence by PMT, \(2 \mid (n^2 + 3n) \quad \forall n \geq 1\).

\(\square\)

\[\begin{align*}
\sum_{i=0}^{n} 3^i &= \frac{3^{n+1} - 1}{2} \\
\text{By PMT,} \\
\sum_{i=0}^{k} 3^i &= \frac{3^{k+1} - 1}{2}
\end{align*}\]

\(\square\)

II. Counting

1. (2 pts) Let A be the set of all bit stings of length 10. Find |A|.

\[\binom{10}{2} \]

2. (2 pts) Let A be the set of all bit stings of length 10. How many bit strings begin with 1101?

\[\frac{1101}{10} \]

3. (2 pts) A club with 20 women and 17 men needs to send a representative to the club counsel. How many ways can a representative be selected?

\[\binom{37}{2} \]