

## Permutations and Combinations

- What do we mean by a permutation of the elements of a set?
- There are a couple of ways to define this concept and we will look at two of them.

But first let us look at an example: Let us consider the elements  $a$ ,  $b$ , and  $c$  as follows:

$(a, b, c)$ ,  $(a, c, b)$ ,  $(b, a, c)$ ,  $(b, c, a)$ ,  $(c, a, b)$ ,  $(c, b, a)$ .

Each is a rearrangement of the three objects  $a$ ,  $b$  and  $c$ .

One way to look at each of the rearrangements is as a one-to-one function. For example:

1.  $(a, b, c)$  is the function  $f : \{1,2,3\} \rightarrow \{a,b,c\}$ , where  $f(1) = a$ ,  $f(2) = b$  and  $f(3) = c$ .
2.  $(a, c, b)$  is the function  $g : \{1,2,3\} \rightarrow \{a,b,c\}$ , where  $g(1) = a$ ,  $g(2) = c$  and  $g(3) = b$ .
3.  $(b, a, c)$  is the function  $h : \{1,2,3\} \rightarrow \{a,b,c\}$ , where  $h(1) = \underline{\hspace{1cm}}$ ,  $h(2) = \underline{\hspace{1cm}}$  and  $h(3) = \underline{\hspace{1cm}}$ .

**Question:** How many rearrangements are there of an  $n$ -element set? (i.e. how many one-to-one functions are there from an  $n$ -element set to another  $n$ -element set?)

More generally, we are interested in the following types of permutations. Given  $n$  objects how many  $r$ -element rearrangements are there of these objects assuming that  $n > r$ .

**Example:** Given the 26 letters in the alphabet, how many 3-letter arrangements are possible, assume no letter is repeated. Some examples of 3-letter arrangements are  $(a, b, c)$ ,  $(a, f, k)$ ,  $(p, b, n)$  and so on.

**Solution:** *Each* 3-letter arrangement is a one-to-one function. For example:  $(a, f, k)$  is the function  $f: \{1, 2, 3\} \rightarrow \{a, b, c, \dots, y, z\}$ , define as  $f(1) = a$ ,  $f(2) = f$ , and  $f(3) = k$ .

Thus there are \_\_\_\_\_ 3-letter arrangements of the alphabet.

Thus on the one hand: An **k-permutation** of an  $m$ -element set with distinct elements,  $S$ , is a one-to-one function  $f: \{1, 2, \dots, k\} \rightarrow S$ .

Another definition: A **k-permutation** of an  $m$ -element set with distinct elements is an ordered arrangement of  $k$  elements of the set.

**Theorem.** The number of  $k$ -permutations of a set with  $m$  distinct elements, where  $0 \leq k \leq m$ , is

$$P(m, k) = (m)_k = \frac{m!}{(m-k)!}$$

**Proof.** Each  $k$ -permutation is a one-to-one function from a  $k$ -element set to an  $m$ -element set. Thus by Example 8 p. 235 (discussed in class), the result follows.

**Def.** A **k-combination** of elements of an  $m$ -element set is an unordered selection of  $k$  elements from the set, i.e. an  $r$ -combination is simply a subset of the set with  $r$  elements.

**Theorem.** The number of  $k$ -combinations of a set with  $m$  distinct elements, where  $0 \leq k \leq m$ , is

$$C(m, k) = \binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

**Proof:** Let  $C(m, k) = n$ . The  $k$ -permutations of a set with  $m$  distinct elements can be obtained by forming the  $n$   $r$ -combinations of the

set, and then ordering the elements in each  $r$ -combinations, which can be done in  $P(k,k)$  ways. Consequently,

$$P(m,k) = C(m,k) P(k,k).$$

This implies that

$$C(m,k) = \frac{P(m,k)}{P(k,k)} = \frac{m!/(m-k)!}{k!/(k-k)!} = \frac{m!}{k!(m-k)!}.$$

**Example:** How many subsets of the alphabet are there? How many of these subsets contain exactly 5 letters?

**Example:** In how many ways can two students be chosen from a class of 18?

**Example:** In how many ways can two students be chosen from a class of 18 if one of them receives an A and the other receives a B?

**Example:** How many ways are there to select 5 players from a 10-member tennis team to make a trip to another school?

**Example:** How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

**Corollary:** Let  $m$  and  $k$  be nonnegative integers with  $k \leq m$  .

$$\text{Then } \binom{m}{k} = \binom{m}{m-k}.$$

Proof:

**Exercise:** What interpretation can we make about sets for the previous corollary?

## **Binomial Coefficients**

**Terminology:** The number  $\binom{n}{r}$  is also called a **binomial coefficient** because they occur as coefficients in the expansion of powers of binomial expressions such as  $(a + b)^n$  .

**Theorem** (Pascal's Identity) Let  $n$  and  $k$  be positive integers with

$$n \geq k \quad . \text{ Then } \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof.

Using Pascal's Identity we can construct Pascal's Triangle

**Theorem (The Binomial Theorem)** Let  $x$  and  $y$  be variables, and let  $n$  be a positive integer. Then

$$(x + y)^n = \sum_{j=0}^n C(n, j)x^{n-j}y^j$$
$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n.$$