Mathematical Induction

Well-Ordering Property (Axiom)
If **A** is any nonempty subset of the set of positive integers, then among the elements of **A** there is a smallest one.

Note: **N** denotes the natural numbers {1, 2, 3, 4, ….}

Theorem. (The Principle of Math Induction) (PMI)
Let $M \subseteq \mathbf{N}$.
If (i) $1 \in M$
(ii) If $k \in M$, then $k+1 \in M$,
then $M = \mathbf{N}$.
Proof. Assume $M \subseteq \mathbf{N}$, $1 \in M$ and that if $k \in M$, then $k+1 \in M$.

By way of contradiction suppose that $M \neq N$. Let us define the following set,

$$S = \mathbf{N} - M.$$

Since the set M is a proper subset of **N**, the set S is nonempty. Since S is a subset of the **N** and nonempty, the Well-Ordering Axiom (WOA) implies that there is a smallest integer x in S. By assumption $1 \in M$, which implies that $1 \notin S$. This in turn implies that the smallest integer x in S is greater that one, i.e. $x > 1$. Since $x \in S$, by definition of S, $x \notin M$. By the contrapositive of (ii), $x \notin M$ implies that $x-1 \notin M$. By construction of S and $x-1 \notin M$, we conclude that $x-1 \in S$. But x – 1 is smaller than x, and thus we have contradicted that x is the smallest element of S.

Hence, $M = \mathbf{N}$.

When to use the Principle of Mathematical Induction:

- When we need to prove a mathematical statement for every natural number.

How to use the Principle of Mathematical Induction:

- **(1)** Identify the math statement to be proven.
  (2) Show that the statement is true for the natural number 1.
  (3) Show that if we assume that the statement is true for some *k*, then it follows that the statement must also be true for *k+1*, i.e. property (ii).
  (4) Conclusion: By the Principle of Math Induction….

Theorem $\forall n \in N, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof: Let ***P(n)*** be the statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and let

$$M = \{n \in N : P(n) \text{ is true}\}.$$

Theorem Let x and y denote any pair of real numbers for which $0 < x < y$. For each $n \in N$, $0 < x^n < y^n$.

Proof. Assume x and y are any pair of real numbers for which $0 < x < y$. Let $P(n)$ denote the statement $0 < x^n < y^n$. By assumption $P(1)$ is true.

Assume $P(k)$ is true for some positive integer k. This means that $0 < x^k < y^k$ for some positive integer k. Since y and x^k are positive, multiplying the inequality $0 < x < y$ by x^k will not reverse the inequality, and we get

$$0 < x^{k+1} < yx^k. \tag{1}$$

Next multiply $0 < x^k < y^k$ by y to get

$$0 < yx^k < y^{k+1}. \tag{2}$$

Combining the results in (1) and (2) using transitivity of inequalities yields,

$$0 < x^{k+1} < yx^k < y^{k+1}.$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true. Hence by PMI, $P(n)$ is true $\forall n \in N$.

QED

Example: Prove that for all $n \in N$, $\left(1+\frac{1}{2}\right)^n \geq 1+n/2$.

Proof: Let *P(n)* ____.

Since $\left(1+\frac{1}{2}\right)^1 = 1+1/2$, *P(1)* is true.

Assume ___________________________. This means that

____.

$$\left(1+\frac{1}{2}\right)^{n+1} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$\geq \left(1+\frac{1}{2}\right)\left(1+\frac{n}{2}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= 1+\frac{1}{2}+\frac{n}{2}+\frac{n}{4} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= 1+\frac{n+1}{2}+\frac{n}{4} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$\geq 1+\frac{n+1}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Thus if *P(n)* is true, then *P(n+1)* is also true. Hence by PMI,
$\left(1+\frac{1}{2}\right)^n \geq 1+n/2$ ______________. QED

Example: Can PMI be used to show that $\forall n \in N, \ \ n = n+1$?

Solution: Let *P(n)* be the statement *n = n +1*. Assume *P(k)* is true, that is assume *k = k +1* for some integer *k*.

k + 1 = (k+1) + 1 since *P(k)* is true
= k + 2.

Thus *P(k+1)* is true whenever *P(k)* is true. Hence by PMI,….?

What happened? How could we prove this nonsense?

Sums of Geometric Progressions. Use mathematical induction to prove the following formula for the sum of a finite number of terms of a geometric progression.

$$\sum_{j=0}^{n} ar^{j} = \frac{ar^{n+1} - a}{r-1}, \qquad \text{when } r \text{ is not equal to } 1.$$

Example: Use mathematical induction to prove that $\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}$, whenever the A_i are subsets of a universal set U and n is greater than 1.