Functions

Def. Let $A$ and $B$ be sets. A $\textbf{function f from A to B}$ is an assignment of exactly one element of $B$ to each element of $A$.

$$\forall a \in A \ \exists! b \in B \ b \text{ is assigned to } a$$

$$\forall a \in A \ \exists! b \in B \ f(a) = b$$

Notation: If $f$ is a function from $A$ to $B$, we write $f: A \rightarrow B$ and then describe the rule for assignment (i.e. the relationship of elements $A$ to elements of $B$).

Example 1:

True or False: $f$ is a function from the set $A=\{1, 2, 3, 4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 1, f(3) = 2, f(4) = 5$.

True or False: $f$ is a function from the set $A=\{1, 2, 3, 4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 4, f(3) = 2, f(4) = 5$.

True or False: $f$ is a function from the set $A=\{1, 2, 3, 4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 4, f(3) = 2, f(4) = 5, f(5) = 6$.

Definition: A bit string is a sequence of 0’s and 1’s. Let us assume that the first bit is the rightmost bit.

Example 2: Determine whether $f$ is a function from the set of all finite length bit strings to the set of integers if

a) $f(S)$ is the position of a 1 bit in the bit string $S$.

b) $f(S)$ is the smallest integer $i$ such that the $i^{th}$ bit of $S$ is 1.

c) $f(S)$ is the smallest integer $i$ such that the $i^{th}$ bit of $S$ is 1 and $f(S) = 0$ whenever $S$ is the empty string or the constant 0’s string.
Example 3: Determine whether \( f \) is a function from the set of real numbers to the set of real numbers if

a) \( f(a) = \sqrt{a} \)

b) \( f(a) = a^2 \)

Definition.
- If \( f \) is a function from \( A \) to \( B \), we say that \( A \) is the domain of \( f \) and \( B \) is the codomain of \( f \).
- The range of \( f \) is the following set \( \{ b \mid \exists a \in A \ f(a) = b \} \) (Note that this is a subset of the codomain).
- Also, if \( f \) is a function from \( A \) to \( B \), we say that \( f \) maps \( A \) to \( B \).

Example 4: Let \( R \) denote the set of real numbers, and let \( Z \) denote the set of integers.
Define \( f:Z \rightarrow R \) by \( f(n) = \frac{n}{2} \).
Which set is the domain of \( f \)?
(a) \( R \)  (b) \( Z \)  (c) the set of even integers

Which set is the codomain of \( f \)?
(a) \( R \)  (b) \( Z \)  (c) the set of even integers

Which set is the range of \( f \)?
(a) \( R \)  (b) the set of rational numbers with denominator 2  (c) the set of even integers

Example 5: Let \( Z \) be the set of integers, \( Z^+ \) be the set of positive integers, and \( Z^+ \cup \{0\} \) denote the set of nonnegative integers. Define \( f:Z \rightarrow Z \) by \( f(x) = x^2 \).

Which set is the codomain of \( f \)?
(a) \( Z \)  (b) \( Z^+ \)  (c) \( Z^+ \cup \{0\} \)

Which set is the range of \( f \)?
(a) \( Z \)  (b) \( Z^+ \)  (c) \( Z^+ \cup \{0\} \)
Special Functions

**Definition:** The floor function assigns to the real number \( x \) is the largest integer that is less than or equal to \( x \). The value of the floor function at \( x \) is denoted by \( \lfloor x \rfloor \).

**Example 6:** \( \lfloor 1.99 \rfloor = \underline{\ _\ } \), \( \lfloor 2.003 \rfloor = \underline{\ _\ } \), \( \lfloor -1.5 \rfloor = \underline{\ _\ } \)

- What is the domain of the floor function?
- What is the range of this floor function?

**Example 7:** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = 2\lfloor x \rfloor \)

- What is the range of \( f \)?
  (a) \( \mathbb{Z} \)
  (b) \( \mathbb{R} \)
  (c) neither \( a \) nor \( b \)

**Definition:** The ceiling function assigns to the real number \( x \) is the smallest integer that is greater than or equal to \( x \). The value of the ceiling function at \( x \) is denoted by \( \lceil x \rceil \).

**Definition:** Let \( A \) be a set. The identity function on \( A \) is the function \( i_A : A \to A \) where \( i_A(x) = x \).

**Properties of Functions**

**Definition.** A function \( f : A \to B \) is said to be one-to-one, or injective, if and only if \( f(x) = f(y) \) implies that \( x = y \) for all \( x \) and \( y \) in the domain of \( f \), that is if and only if \( \forall x \in A \forall y \in A (f(x) = f(y) \to (x = y)) \).

**Example 8:**

**True or False:** The function \( f \) from \( \{a, b, c, d\} \) to \( \{1, 2, 3, 4, 5\} \) with \( f(a) = 4, f(b) = 5, f(c) = 1, \) and \( f(d) = 3 \) is one-to-one.

**True or False:** The function \( f \) from \( \{a, b, c, d\} \) to \( \{1, 2, 3, 4, 5\} \) with \( f(a) = 4, f(b) = 4, f(c) = 1, \) and \( f(d) = 3 \) is one-to-one.

**True or False:** The function \( f(x) = x^2 \) from the set of integers to the
set of integers is one-to-one.

**True or False:** Let $S$ be a bit string of length $n$. If we define $f(S)$ as the smallest integer $i$ such that the $i^{th}$ bit of $S$ is 1 and $f(S) = 0$ when $S$ is the empty string, then $f$ one-to-one.

**Observation:**
A function $f:A \rightarrow B$ is one-to-one if and only if $\forall x \forall y (f(x) = f(y) \rightarrow x = y)$, which is logically equivalent to its contrapositive $\forall x \forall y (x \neq y \rightarrow f(x) \neq f(y))$.

**Example 9:** Prove that the real valued function $f(x) = x + 1$ is one-to-one.

**Definition.** A function $f$ is said to be **onto**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

$\forall b \in B \ \exists a \in A \ f(a) = b$

**Example 10:**

**True or False:** The function $f$ from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$ is an onto function.

**Example 11:** Determine whether the real valued function $f(x) = x + 1$ is onto.

**Example 12:** Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is onto.
Definition. The function \( f \) is called a **one-to-one correspondence**, or a **bijection** if it is both one-to-one and onto.

**Definition.** Let \( f \) be a bijection from the set \( A \) to the set \( B \). The **inverse function of** \( f \) is the function that assigns to an element \( b \) belonging to \( B \) the unique element \( a \) in \( A \) such that \( f(a) = b \). The inverse function of \( f \) is denoted by \( f^{-1} \). Hence, \( f^{-1}(b) = a \) when \( f(a) = b \).

**Terminology:** A one-to-one correspondence is called **invertible**.

**Example 13:** Let \( f \) be the function from \( \{a, b, c\} \) to \( \{1, 2, 3\} \) defined by \( f(a) = 3 \), \( f(b) = 2 \), \( f(c) = 1 \). Is \( f \) an invertible function? If so describe the inverse function.

**Example 14:** Let \( f : Z \to Z \) defined by \( f(x) = x + 1 \). Is \( f \) invertible? If so describe the inverse function.

**Example 15:** Let \( f : Z \to Z \) defined by \( f(x) = x^2 \). Is \( f \) invertible? If so describe the inverse function.
**Definition.** Let \( g \) be a function from the set \( A \) to the set \( B \) and let \( f \) be a function from the set \( B \) to the set \( C \). The **composition of the functions** \( f \) and \( g \), denoted by \( f \circ g \), is defined by \( (f \circ g)(x) = f(g(x)) \).

**Example 16:** Let \( f : Z \to Z \) and \( g : Z \to Z \) defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \). What is the composition of \( f \) and \( g \)? What is the composition of \( g \) and \( f \)?

**Fact:** Let \( f : A \to B \) be an invertible function. If \( f(a) = b \), then
\[
(f^{-1} \circ f)(a) = a \quad \text{and} \quad (f \circ f^{-1})(b) = b.
\]

**Graphs of Functions**

**Definition.** Let \( f : A \to B \). The **graph of the function** \( f \) is the set of ordered pairs \( \{(a, b) \mid a \in A \text{ and } f(a) = b\} \) (A graph is a subset of the Cartesian product \( A \times B \)).

**Example 17:** Let \( f : Z \to Z \) be defined by \( f(n) = 2n + 1 \). Display the graph of \( f \).

**Example 18:** Let \( f : R \to Z \) be defined by \( f(x) = \lfloor x \rfloor \). Display the graph of \( f \).

**Example 19:** Let \( f : Z \to Z \) be defined by \( f(x) = \lfloor \frac{x}{3} \rfloor \). Display the graph of \( f \).