Computation of Nonclassical Shocks Using a Spacetime Discontinuous Galerkin Method

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ABSTRACT

We present a numerical study for two systems of conservation laws using a spacetime discontinuous Galerkin (SDG) method with causal spacetime triangulations and the piecewise constant Galerkin basis. The SDG method is consistent with the weak formulation of conservation laws, and, in the case of strictly hyperbolic systems, also with the Lax entropy condition. Convergence of the method was shown for a special class of hyperbolic systems (Temple systems).

The initial data we consider lead to nonclassical shocks. The first part of our study is for the Keyfitz-Kranzer system. We compute the SDG solutions approximating overcompressive and singular shocks, and note that our results are consistent with those obtained by [Sanders, and Sever 2003] using a finite difference scheme. The second system we consider is an approximation of a three-phase flow in the petroleum reservoirs. Numerical solutions for this system were computed by [Schecter, Plohr, and Marchesin 2004] using the Dafermos regularization and a technique for numerical solving of ordinary differential equations. We compute the SDG approximation to a solution containing a transitional shock.

We note that even though convergence of the SDG method was shown so far only for Temple systems, numerical examples herewith show that it can be successfully used in approximating solutions of more general conservation laws.

Categories and Subject Descriptors

G.1.8 [**Numerical Analysis**]: Partial Differential Equations—*finite element methods, hyperbolic equations*

General Terms

Performance of a numerical method

Keywords

conservation laws, discontinuous Galerkin, shocks

1. INTRODUCTION

Conservation laws describe various phenomena involving advective transport or wave motion, as gas dynamics, elastodynamics, biomechanics, chromatography, oil recovery, etc. The study of these partial differential equations is very challenging because they poses some special features that are not seen elsewhere in the PDE theory. Solutions may become discontinuous after a finite time even for smooth initial data and one has to define weak solutions. Weak solutions are not necessarily unique, and to distinguish a physically meaningful solution additional conditions are needed (see [Lax 1973], [Serre 1999], [Dafermos 2000]).

The development of numerical methods for conservation laws involves an interplay between physical modeling, mathematical theory and numerical analysis. Finite difference methods (for eg., [Lax, and Wendroff 1964], [Glimm 1965], [Osher 1984]) and finite volume methods (for eg., [LeVeque 2002]) are the most studied and widely used numerical methods for conservation laws. In finite difference methods, the solution is approximated pointwise at the grid points, while the finite volume methods are based on the integral form of conservation laws instead, and one is approximating the cell average of the exact solution. Discontinuous Galerkin methods (see [Cockburn, Karniadakis, and Shu 2000]) use an element-wise representation of a solution and enforce the conservation law locally. The Runge-Kutta DG methods ([Cockburn 2001]) are based on a finite element discretization of a spatial domain and a special Runge-Kutta type discretization is used to propagate the solution in time.

In this paper we consider a DG method based on spacetime discretizations.

2. NUMERICAL STUDY OF CONSERVATION LAWS USING THE SDG METHOD

Consider an initial value problem for a one-dimensional system of conservation laws

$$\begin{aligned} &u_t + f(u)_x = 0, \quad (t,x) \in [0,\infty) \times \mathbf{R}, \\ &u(0,x) = u_-(x), \end{aligned} \tag{1}$$

where $u : [0,\infty) \times \mathbf{R} \to \mathcal{D} \subseteq \mathbf{R}^n$ denotes the vector of densities of conserved variables (such as mass, momentum, energy), $f : \mathcal{D} \to \mathbf{R}^n$ is the spatial flux, and $u_- : \mathbf{R} \to \mathcal{D}$ is a function of bounded total variation. Let Df denote the gradient matrix of f and let λ^i and r^i denote the i-th eigenvalue and the i-th right eigenvector of Df, respectively.

Given $u \in \mathcal{D}$, if $\lambda^1(u), \ldots, \lambda^n(u)$ are real and distinct, the system (1) is *strictly hyperbolic* in the domain of conservation states $\mathcal{D}$. If for each $i \in \{1, \ldots, n\}$, λ^i is strictly monotone along the i-th integral curve, meaning $D\lambda^i(u) \cdot r^i(u) \neq 0$, $u \in \mathcal{D}$, the system is *genuinely nonlinear* in $\mathcal{D}$.

2.1 Definition of the SDG Method

The spacetime discontinuous Galerkin method is a finite element method based on spacetime partitions of the domain $[0, \infty) \times \mathbf{R}$. The Galerkin basis consists of functions which are polynomials of a fixed degree $k \geq 0$ within each spacetime element, but might be discontinuous across element boundaries. The values of approximants on adjacent elements are coupled through the Godunov flux. The method can be used on both layered and unstructured grids. A direct element-by-element solution procedure is possible, and the computational complexity of the method is $O(N)$, where N denotes the number of elements within the mesh.

The SDG method was first introduced in [Palaniappan, Haber, and Jerrard 2004] with numerical examples for scalar hyperbolic conservation laws. The method considered in [Lowrie 1996] is similar to the SDG method, but is based on uniform layered spacetime grids and uses an approximation of the Godunov flux on certain element boundaries.

Analysis of the SDG method is rather challenging due to spacetime partitions and coupling of values of approximate functions via the Godunov flux. As in [Jegdic 2004], throughout this work we assume that the domain partitions are triangulations, and we impose two simplifying assumptions:

(a) if $\mathcal{T}_h$ is a spacetime triangulation (h stands for the maximal diameter of an element in the considered triangulation), then for each edge Γ of an element $T \in \mathcal{T}_h$ with the outward unit spacetime normal $\nu = (\nu_t, \nu_x)$, we require that

$$\text{either} \qquad (1, \lambda^i(u))\,\nu < 0 \qquad \text{or} \qquad (1, \lambda^i(u))\,\nu > 0, \tag{2}$$

for every $i \in \{1, \ldots, n\}$ and all $u \in \mathcal{D}$, and

(b) the Galerkin basis consists of piecewise constant functions, i.e., $k = 0$. Given a spacetime triangulation $\mathcal{T}_h$, we denote the corresponding Galerkin basis by $\mathcal{P}_h$.

The condition (2) is called the *causality constraint*. Given an edge Γ of an element $T \in \mathcal{T}_h$ with the outward spacetime normal ν, if the expression $(1, \lambda^i(u))\,\nu$ is negative (positive) for all $i \in \{1, \ldots, n\}$ and $u \in \mathcal{D}$, than the edge Γ is said to be *inflow (outflow)* for T.

Then, the formulation of the *causal spacetime discontinuous Galerkin method* is:

Given a causal spacetime triangulation $\mathcal{T}_h$ of the domain $[0, \infty) \times \mathbf{R}$, find $u_h \in \mathcal{P}_h$ such that

$$\int_{\partial T^-} (u_h^-, f(u_h^-)) \cdot \nu \, d\mathcal{H}^1 + \int_{\partial T^+} (u_h, f(u_h)) \cdot \nu \, d\mathcal{H}^1 = 0, \tag{3}$$

holds on each element $T \in \mathcal{T}_h$. Here, ∂T^- and ∂T^+ stand for the inflow and outflow part of the boundary ∂T, respectively, u_h^- denotes the value of the approximant along ∂T^- which is computed on an adjacent mesh element, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

In the case of strictly hyperbolic systems of conservation laws, we show in [Jegdic 2004] and [Jegdic, and Jerrard 2004] that given a causal spacetime triangulation $\mathcal{T}_h$, if a SDG approximation $u_h \in \mathcal{P}_h$ exists, then it must satisfy certain discrete entropy inequalities. These entropy inequalities are discretized versions of the Lax entropy-entropy flux condition in the context of the SDG method. Furthermore, in the case of strictly hyperbolic genuinely nonlinear Temple systems, as introduced in [Temple 1983], we show that given a sequence of causal spacetime triangulations $\{\mathcal{T}_h\}$, the corresponding sequence $\{u_h\}$ of SDG approximations exists and is precomact in $L^1_{loc}([0, \infty) \times \mathbf{R}; \mathbf{R}^n)$. The main point in the proof is to show local Riemann invariant bounds which relies heavily on the structure of Temple systems. Once this local property is established, we show that any limit of any convergent subsequence of $\{u_h\}$ is a weak solution to the initial value problem (1).

2.2 Numerical Study of the Keyfitz-Kranzer System

The first three examples of our numerical study are for the Keyfitz-Kranzer system

$$\begin{aligned} &(u_1)_t + (u_1^2 - u_2)_x = 0, \\ &(u_2)_t + (\tfrac{1}{3}u_1^3 - u_1)_x = 0. \end{aligned} \tag{4}$$

(For more details, see [Keyfitz, and Kranzer 1995].) This system is strictly hyperbolic and genuinely nonlinear in $\mathbf{R}^n$ with eigenvalues $\lambda^1(u) = u_1 - 1$ and $\lambda^2(u) = u_1 + 1$. It is known that for some choices of the initial data there is no solution to (4) consisting of classical rarefaction waves and shocks, and that the candidates for solutions include distributions.

We consider the initial data studied in [Sanders, and Sever 2003] using a finite difference scheme. These data result in overcompressive and singular shocks. A change of coordinate $x \mapsto \bar{x} := x - st$ is performed to ensure that the computed shocks are nonmoving (here, s denotes the shock speed determined from the initial data). For each example we use a layered triangulation, denoted by $\mathcal{T}$, which is refined around $\bar{x} = 0$. The size of a triangle $T \in \mathcal{T}$ is described by two parameters

$$\Delta t := \max_{(t,x)\in T} t - \min_{(t,x)\in T} t \text{ and } \Delta x := \max_{(t,x)\in T} x - \min_{(t,x)\in T} x. \tag{5}$$

We specify Δt and the interval for Δx, so that the causality constraint (2) is satisfied on all triangles $T \in \mathcal{T}$. Examples 1 and 2 present the SDG approximations to (4) with Riemann initial data

$$u_-(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0. \end{cases} \tag{6}$$

Example 1. Consider the Riemann problem (4), (6) with $u_l = (1.5, 0)$ and $u_r = (-1.895644, 1.343466)$. We compute

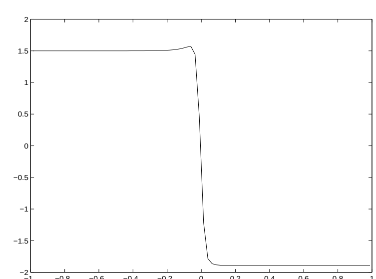

Figure 1: The SDG approximation of u_1.

the SDG approximation on a layered causal triangulation with $\Delta t = 0.0025$ and $\Delta x \in [0.0075, 0.025]$. The approximations to u_1 and u_2 at time $t = 4$ are depicted in Figures

1 and 2. This type of solution to (4) is known as an overcompressive shock.

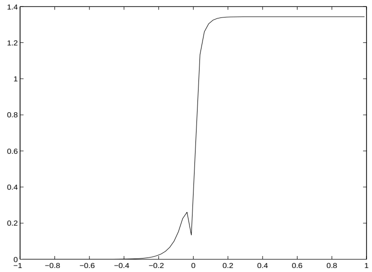

Figure 2: The SDG approximation of u_2.

Example 2. We consider (4), (6) with $u_l = (1.5, 0)$ and $u_r = (-2.065426, 1.410639)$. The approximation is computed at the time $t = 4$ on a causal layered triangulation with $\Delta t = 0.002$ and $\Delta x \in [0.008, 0.02]$, where Δt and Δx are defined as in (5). This type of a solution to the Keyfitz-Kranzer system is known as a singular shock and the second component, u_2, of the solution is unbounded (Figures 3, 4).

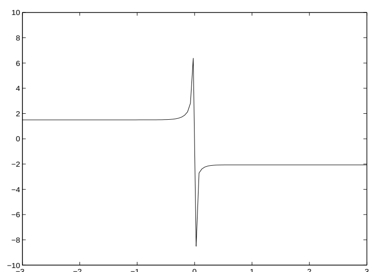

Figure 3: The SDG approximation of u_1.

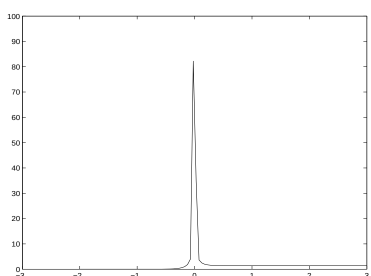

Figure 4: The SDG approximation of u_2.

Example 3. In our last example for the Keyfitz-Kranzer system we consider the smooth initial data of the form

$$u_-(x) = \frac{u_l + u_r}{2} - \frac{u_l - u_r}{4}\, x\,(3 - x^2),$$

where u_l and u_r are as in Example 2. The second component of this initial data is depicted in Figure 5. The SDG approximations shown in Figures 6, 7 and 8 are computed for the second component u_2 of the solution which becomes unbounded. They are computed at times $t = 0.3$, 0.6 and 0.9, on a triangulation with $\Delta t = 0.001$ and $\Delta x \in [0.005, 0.01]$.

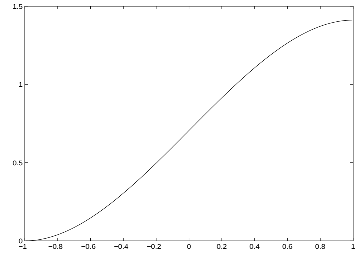

Figure 5: The initial data for the component u_2.

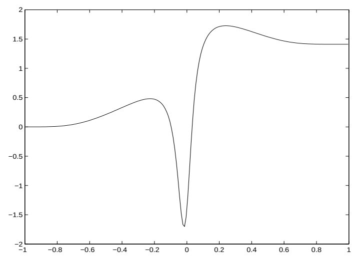

Figure 6: The SDG approximation of u_2 at $t = 0.3$.

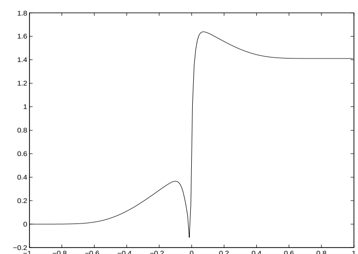

Figure 7: The SDG approximation of u_2 at $t = 0.6$.

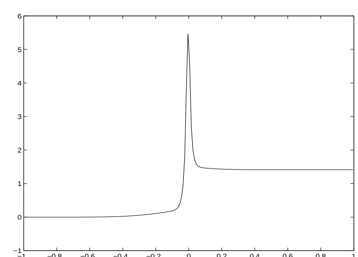

Figure 8: The SDG approximation of u_2 at $t = 0.9$.

2.3 Numerical Study of an Approximation to a Three Phase Flow

In the second part of our numerical study of nonclassical shocks we consider the following system of conservation laws

$$\begin{aligned}&(u_1)_t + (-0.5\,u_1^2 + 0.5\,u_2^2 - 0.12\,u_1 + 0.23\,u_2)_x = 0,\\ &(u_2)_t + (u_1\,u_2 - 0.23\,u_1 - 0.12\,u_2)_x = 0.\end{aligned} \qquad (7)$$

This system is of mixed type with eigenvalues given by

$$-0.12 \pm \sqrt{u_1^2 + u_2^2 - 0.529}.$$

We find the SDG approximation of the solution for one of the Riemann problems numerically studied in [Schecter, Plohr, and Marchesin 2004]. Using the Dafermos regularization and change of variables $(x,t) \mapsto \xi := x/t$, these authors obtained a system of ordinary differential equations in variable ξ which they numerically solved using the AUTO continuation technique. The Riemann initial data in this last example is given by

$$u_l = (0.366078, 0.308156) \quad \text{and} \quad u_r = (-0.61, 0.1).$$

The SDG solution is computed at the time $t = 2$ on a layered causal triangulation with parameters $\Delta t = 0.005$ and $\Delta x \in [0.005, 0.01]$. We note that initial data implies that the eigenvalues become complex, and a solution consists of a 1-shock, 1-transitional shock and a composite 2-wave (a 2-transitional shock plus a 2-rarefaction). The SDG approximations for both components, u_1 and u_2, of this solution are depicted in Figures 9 and 10, respectively.

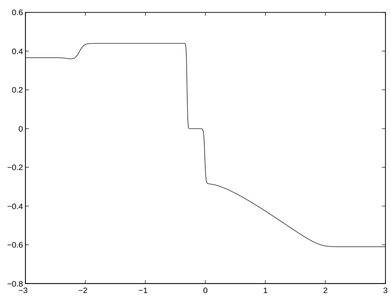

Figure 9: The SDG approximation of u_1.

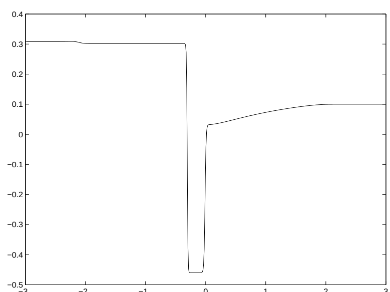

Figure 10: The SDG approximation of u_2.

3. CONCLUSIONS

We remark that the SDG approximations obtained in all of the above examples are consistent with numerical solutions in [Sanders, and Sever 2003] and [Schecter, Plohr, and Marchesin 2004].

We note again that convergence of the SDG method was proved only for a special class of hyperbolic systems (Temple systems) using their special geometric structure. However, the numerical experiments presented in this paper show that the SDG method can be successfully used in approximating solutions to more general systems of conservation laws.

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