Projective Fourier Analysis in Computer Vision: Theory and Computer Simulations

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SPIE Vol. 3168, pp.124-135, 1997

ABSTRACT

Identifying the projective group for patterns by developing the camera model, the projective Fourier transform and its inverse are obtained in analogy with the classical, that is, Euclidean Fourier analysis. Projectively adapted properties are demonstrated in a numerical test. Using the expression of the projective Fourier integral by a standard Fourier integral in the coordinates given by the complex principal logarithm, the discrete projective Fourier transform and its inverse are constructed showing that FFT algorithms can be adapted for their computations.

Keywords: Projective Fourier transform, projectively adapted pattern representation, discrete projective Fourier transform, fast projective Fourier transform algorithms.

1 Introduction

It is well known that *one can reconstruct any rotated and translated image using the only one Fourier transform of the original image.* When a perspective projection is applied, the relationship between Fourier transforms of the original image and its distorted images is no longer feasible. However, a representation of images such that one has a closed-form relation between the representations of the original image and its projective distortions is an important step in developing a system for automated perspective-independent object recognition.

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Motivated by this deficiency of the Fourier analysis, we have developed in[3] its projective analogue for patterns. The projectively adapted characteristics of this harmonic analysis were confirmed by a numerical test in which projective distortions of a circular ring were reconstructed using only the projective Fourier transform of the ring pattern.

In this paper, following[4] , we develop the discrete projective Fourier transform of a pattern and show that the FFT can be adapted to its computation. The explanation of this surprising fact follows from the analogy of the projective harmonic analysis with the classical (Euclidean) harmonic analysis. Briefly, the projective Fourier transform is defined using characters of the abelian subgroup (isomorphic with the multiplicative group of nonzero complex numbers $\mathbb{C}_*$) of pattern's projective group $\mathbf{SL}(2, \mathbb{C})$. However, these characters can be extended to a subgroup of $\mathbf{SL}(2, \mathbb{C})$ that exhausts almost all projective distortions of a pattern.

This simpler aspect of the analysis will be stressed in this article. However, the projective Fourier analysis can only be fully understood in terms of the representation theory of semisimple Lie group $\mathbf{SL}(2, \mathbb{C})$, see[3] , where also detailed exposition of related projective geometry has been given.

2 The group of projective transformations for patterns

2.1 A pinhole camera

We start with the description of a pinhole camera. The pinhole, or optical center, of the camera is the point where the incoming rays of light intersect each other, giving an image on the image plane. The line passing through the optical center and perpendicular to the image plane is called the optical axis of the camera.

In order to formulate a camera model quantitatively, we consider an image plane to be the plane $x_2 = 1$ in $\mathbb{R}^3 = \{(x_1, x_2, x_3)^t : x_i \in \mathbb{R}\}$. The image of $(x_1, x_2, x_3)^t$ on the image plane is given by the projection $j : \mathbb{R}^3 \to \mathbb{C}$ defined by

$$j \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{x_3 + ix_1}{x_2} \tag{1}$$

where we have identified $(x_1, 1, x_3)^t$ in the image plane with $z = x_3 + ix_1$.

We restrict our discussion to *patterns*, that is, planar objects. Assuming that patterns *"live"* on the image plane $x_2 = 1$, the projective distortions of a pattern are generated by translating or rotating the pattern to form its *"virtual"* space position and then projecting by (1) on the image plane, and by all finite iterations of these basic distortions. We denote the set generated by all such iterations by $\mathbf{G}^{\times}$ and we find it in (10) in Section 2.3.

2.2 The image plane as the complex projective line

The image plane can be regarded as the extended complex line $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with its affine piece $\mathbb{C}$ given by the equation $x_2 = 1$ if we require that

$$j \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = \infty.$$

Further, if we take

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1 = x_2 + iy,\ z_2 = x_3 + ix_1 \right\}$$

then in the affine patch on $\widehat{\mathbb{C}}$ given by $z_1 = 1$, the points $\xi \neq \infty$ of $\widehat{\mathbb{C}}$ are identified with the points where the lines $z_2 = \xi z_1$ intersect the line $z_1 = 1$, i.e., $\xi = x_3 + ix_1$. The point ∞ corresponds to the line $z_1 = 0$. It shows that $\widehat{\mathbb{C}}$ can be identified with the complex projective line $P^1(\mathbb{C})$ where, roughly speaking,

$$P^1(\mathbb{C}) = \left\{ \text{lines (rays) of } \mathbb{C}^2 \text{ through the origin} \right\}.$$

The group $\mathbf{SL}(2,\mathbb{C})$, consisting of all 2×2 complex matrices of determinant 1, acts on nonzero column vectors $(z_1, z_2)^t \in \mathbb{C}^2$. In the affine patch on $\widehat{\mathbb{C}}$ given by $z_1 = 1$, one can easily check that the action of the element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C})$$

on $\mathbb{C}^2$ induces the action on slopes of the lines $z_2 = \xi z_1$ given by $\xi \longmapsto \xi' = (d\xi + c)/(b\xi + a)$.

Consequently, $\mathbf{SL}(2, \mathbb{C})$ acts on the image plane $\widehat{\mathbb{C}}$ by linear-fractional transformations given by

$$\mathbf{SL}(2, \mathbb{C}) \ni g : z \longmapsto g \cdot z = \frac{dz + c}{bz + a}. \tag{2}$$

and the *projective group* for patterns is the quotient

$$\mathbf{PSL}(2, \mathbb{C}) = \mathbf{SL}(2, \mathbb{C}) / \pm Id \tag{3}$$

since both matrices $\pm g$ give the same orbit under the map (2).

2.3 The projective camera for patterns

The subgroup

$$\mathbf{SU}(2) = \left\{ \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

is the maximal compact subgroup in $\mathbf{SL}(2, \mathbb{C})$.

One can show that for a rotation $r \in \mathbf{SO}(3)$ there are two elements $k \in \mathbf{SU}(2)$ such that $jr = kj$ where j is given in (1). Indeed, if

$$k = \pm \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix}$$

then

$$k \cdot z = \frac{az + b}{-\overline{b}z + \overline{a}} \tag{4}$$

where

$$a = \pm \cos\left(\frac{\phi}{2}\right) e^{i(\psi_1+\psi_2)/2} \quad \text{and} \quad b = \pm i \sin\left(\frac{\phi}{2}\right) e^{i(\psi_1-\psi_2)/2} \tag{5}$$

in which ψ_1, ϕ and ψ_2 are the Euler angles parametrizing $r \in \mathbf{SO}(3)$ such that ψ_1 is the first rotation angle about the x_2-axis, ϕ is the following rotation angle about the rotated x_1-axis, and finally, ψ_2 is the last rotation angle about the x_2-axis, rotated by the previous two angles. This choice implies that ψ_i give the rotations in the image plane $x_2 = 1$ and ϕ gives the rotation out of the image plane.

For translations $b = (b_1, b_2, b_3)^t \in \mathbb{R}^3$, there are also two elements

$$h = \pm \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C})$$

acting by

$$h \cdot \xi = \frac{\alpha^{-1}\xi + \beta}{\alpha} \tag{6}$$

such that $jb = hj$ where j is given in (1), α is given by

$$\alpha = (1 + b_2)^{1/2} \quad \text{if} \quad 1 + b_2 > 0 \tag{7}$$

and by

$$\alpha = i\left(|1 + b_2|\right)^{1/2} \quad \text{if} \quad 1 + b_2 < 0, \tag{8}$$

and finally,

$$\beta\alpha = (b_3 + ib_1). \tag{9}$$

A simple implication of the last result is the factorization of h. We have two cases:

(1) If $1+b_2>0$, then

$$h \in \mathbf{A}\overline{\mathbf{N}}$$

(2) If $1+b_2<0$, then

$$h = \varepsilon\mathbf{A}\overline{\mathbf{N}}, \qquad \varepsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

where

$$\overline{\mathbf{N}} = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \mathbb{C} \right\}$$

and

$$A = \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} : \delta \in \mathbb{R}_+ \right\}.$$

Now, it follows from (4), (6) and the choices of the image plane and Euler angles that the subgroup $\mathbf{SU}(2)$ acting by linear-fractional transformations on the image plane $\mathbb{C}$ by

$$z \longmapsto \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix} \cdot z = \frac{az+b}{-\overline{b}z+\overline{a}}$$

where a and b are given in (5), represents *projective distortions produced by rotations of a camera* with the maximal torus in $\mathbf{SU}(2)$,

$$\mathbf{M} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\},$$

representing the rotations of the camera about the optical axis.

Also, the linear-fractional transformations given by $\mathbf{A}$ (or given by $\varepsilon\mathbf{A}$),

$$z \longmapsto \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \cdot z = \delta^{-2} z$$

where δ is given by α in (7) (or by α in (8)), represent *distortions produced by translations of a camera in the direction of its optical axis.* Finally, the group $\overline{\mathbf{N}}$ acting by the linear-fractional transformations on $\mathbb{C}$ by

$$z \longmapsto \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \cdot z = z + \gamma$$

where γ is given by $\beta\alpha$ in (9), represents *translations of a camera in the direction perpendicular to the optical axis.*

Now, it follows from the factorization

$$\mathbf{SL}(2,\mathbb{C}) = \mathbf{SU}(2)\mathbf{A}\mathbf{SU}(2)$$

see Theorem 5.20 on p. 126 in[1] , that all finite iterations of $\mathbf{SU}(2)$ and $\mathbf{A}$ generate the group $\mathbf{SL}(2,\mathbb{C})$. Recalling (3), we conclude the projective camera model of Section 2.1 with the result:

$$\mathbf{G}^{\times} = \mathbf{PSL}(2,\mathbb{C}). \tag{10}$$

Thus, for a given pattern

$$P = \{f : D \to \mathbb{R}\} \tag{11}$$

with a bounded $D \subset \mathbb{C}$, the orbit

$$\mathbf{PSL}(2,\mathbb{C}) \ni g \longmapsto P_g = \{fg^{-1} : gD \to \mathbb{R}\} \tag{12}$$

provides *projective classification* of images as it exhausts all of the projectively distorted patterns of P and that two patterns that are on different orbits are not related by a projective transformation.

3 The projective Fourier transform

Here we give a simple development of the projective Fourier transform of square integrable functions using characters and stressing the role played by the subgroups of the universal double cover $\mathbf{SL}(2,\mathbb{C})$ of the projective group $\mathbf{PSL}(2,\mathbb{C})$. A fuller account, including representation theory and the related projective geometry, has been given in[3] .

The finite-dimensional irreducible unitary representations of the Borel subgroup $\mathbf{B} = \mathbf{MAN} \subset \mathbf{SL}(2,\mathbb{C})$, where $\mathbf{N} = \overline{\mathbf{N}}^t$, are one-dimensional. Indeed, for $b \in \mathbf{B}$, the representation $T^{k,is}$ is acting on the one-dimensional Hilbert space $\mathbb{C}$ by $T^{k,is}(b)z = \pi_{k,s}(b)z$ where

$$\pi_{k,s}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \left(\frac{a}{|a|}\right)^k |a|^{is} \tag{13}$$

with $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, and every such representation has this form (cf.[2]).

These representations can be obtained by extending to $\mathbf{B}$ the characters $\pi_{k,s} : \mathbf{MA} \to \mathbb{T}$, where $\mathbb{T}$ is the circle group ($\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$), that are given as in (13) but with the matrix element $b = 0$, see[2] . Also, since the (abelian) subgroup $\mathbf{MA}$ is topologically isomorphic with $\mathbb{C}_*$, the multiplicative group of complex numbers, we have that $\pi_{k,s}$ in (13) are also characters of the group $\mathbb{C}_*$, that is,

$$\pi_{k,s}(z) = \left(\frac{z}{|z|}\right)^k |z|^{is}, \quad k \in Z \text{ and } s \in R. \tag{14}$$

It follows from the Gauss factorization

$$\mathbf{SL}(2,\mathbb{C}) \doteq \overline{\mathbf{N}}\mathbf{MAN}, \tag{15}$$

where "$\doteq$" means that the equality holds up to a lower dimensional subset, that $\mathbf{B} = \mathbf{MAN}$ exhausts essentially the projective part of $\mathbf{SL}(2,\mathbb{C})$, because $\overline{\mathbf{N}} \cong \mathbb{C}$ describes translations in the image plane.

Moreover, the resulting projective Fourier transform is extended, by the corresponding Plancherel's theorem, see[3] , to an unitary transformation on the Hilbert space $L^2(\mathbb{C})$ with the norm $||f||_2 = \left(\frac{i}{2}\int |f(z)|^2 dz d\overline{z}\right)^{1/2}$ where if $z = x+iy$ then $\frac{i}{2}dzd\overline{z} = dxdy$. Thus, lower-dimensional sets have measure zero and therefore statements like (15) are allowable.

The classical (Euclidean) Fourier transform on $L^2(\mathbb{R}^2)$ is defined in terms of the characters $\gamma_k(x) = e^{ik\cdot x}$ of the translation (abelian) subgroup $\mathbb{R}^2$ of Euclidean group $\mathbf{SO}(2)\dot{\times}\mathbb{R}^2$. In analogy with this classical case, taking a pattern (11) with a compact support $D \subset \mathbb{C}_*$, the *projective Fourier transform* $\widehat{f}(k,s)$ of $f(z)$ is defined in terms of the characters (14) of the multiplicative group $\mathbb{C}_*$, that is, in terms of the irreducible unitary representations (13) of $\mathbf{B}$, by

$$\widehat{f}(k,s) = \frac{i}{2}\int f(z)\left(\frac{z}{|z|}\right)^{-k}|z|^{-is-1}dzd\overline{z}. \tag{16}$$

The extra factor $|z|^{-1}$ in (16) unitarizes the transformation.

It can be proved (see,[3]) that the *inverse projective Fourier transform,* is

$$f(z) = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(k,s)\left(\frac{z}{|z|}\right)^{k}|z|^{is-1}ds. \tag{17}$$

It gives the decomposition of the pattern in terms of the characters (14) with the coefficients of the decomposition given by the projective Fourier transform $\widehat{f}(k,s)$ given in (16).

We summarize the analysis given here as follows: *The above projective Fourier decomposition is the decomposition in terms of the simplest functions* (13)**,** that is, the irreducible unitary representations of $\mathbf{B}$, *that mirror the projective subgroup* $\mathbf{B}$ *operations. Recall that by the Gauss factorization (*15*),* $\mathbf{B}$ essentially differs from the projective group $\mathbf{SL}(2,\mathbb{C})$ by (unimportant) translations in the image plane.

We conclude that the projective distortions in (12) can be obtained using the only one projective Fourier transform of it as follows

$$\begin{aligned} T_g f(z) &= f(g^{-1}\cdot z) \\ &= (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(k,s)\left(\frac{g^{-1}\cdot z}{|g^{-1}\cdot z|}\right)^{k}|g^{-1}\cdot z|^{is-1}ds. \end{aligned} \tag{18}$$

4 The numerical test

This test (see[3]) demonstrates the projectively adapted properties of the projective Fourier transform. Here, the circular ring pattern is reconstructed using (17) in Figure 1.

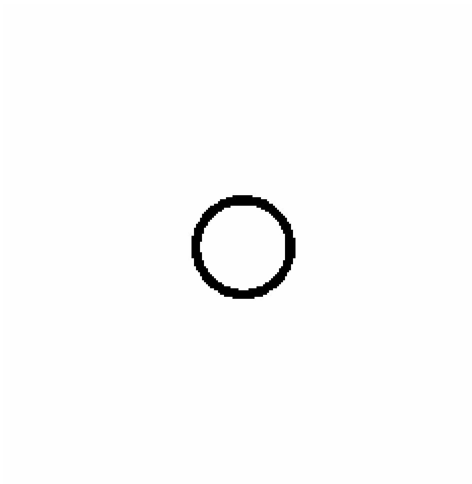

Figure 1

The pattern is next projectively distorted by applying

$$g = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

which represents rotation (out of the image plane $x_2 = 1$) by the angle ϕ about x_1-axis of the ring pattern and projecting it back on the image plane. Computer simulations of (18) with

$$g^{-1}z = \frac{z\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}}{-iz\sin\frac{\phi}{2} + \cos\frac{\phi}{2}}, \qquad (19)$$

are shown in Figure 2 for $\phi = 21.6°$ and in Figure 3 for $\phi = 30°$.

Figure 2

Figure 3

Thus, we have produced the pattern's projective distortions from the only one projective Fourier transform of the original pattern, which confirm projectively adapted characteristics of the image analysis based on the projective Fourier transform.

In the remaining part of this presentation we develop the two steps: the projective Fourier series and the discrete projective Fourier transform with its inverse, leading to the implementation of the fast Fourier transform algorithms in calculating the projective Fourier transform. For a full exposition and implementations of algorithms for fast calculations of projective Fourier transforms, including also discrete projective convolutions, see[4] .

5 The projective Fourier series

By introducing the parametrization by the polar coordinates (r, θ), $r > 0$ and $0 \leq \theta < 2\pi$, given by $z = re^{i\theta}$ into (16) and writing $r = e^u$ we obtain after a simple calculation

$$\widehat{f}(k,s) = \int_{-\infty}^{\infty}\int_{0}^{2\pi} f(e^{u+i\theta})e^u e^{-i(k\theta+us)}d\theta du, \tag{20}$$

which is the projective Fourier transform of f expressed by the standard Fourier transform in the coordinates (u, θ).

In order to derive the projective Fourier series we need to *extend periodically* the function

$$g(u, \theta) = f(e^{u+i\theta})e^u$$

in the u-coordinate ($g(u,\theta)$ is periodic in the θ-coordinate).

To do this, we replace the domain D of a pattern $P = \{f : D \to \mathbb{R}\}$ by a sector $D_s = [0,T] \times \left[0, \frac{2\pi}{L}\right]$, scaling the pattern if necessary, and extend periodically the function $D_s \ni (u,\theta) \longmapsto g(u,\theta)$ in the u-coordinate (using the same name for the extension) by $g(u+mT,\theta) = g(u,\theta)$. This implies the condition on the extension of f,

$$f(e^{u+mT+ik\theta}) = f(e^{u+ik\theta})e^{-mT} \tag{21}$$

where $m \in \mathbb{Z}$.

Now, the projective Fourier transform (16) can be written as

$$\widehat{f}(k,s) = \int_0^T \int_0^{2\pi/L} g(u,\theta)e^{-i(k\theta+su)}d\theta du. \tag{22}$$

Next, we define the function h by

$$h(\vartheta,\gamma) = g(u,\theta) = g(\frac{\vartheta T}{2\pi},\frac{\gamma}{L})$$

where the new variables ϑ and γ are given by $\vartheta = \frac{2\pi u}{T}$ and $\gamma = L\theta$, which is 2π-periodic w.r. to both variables, and therefore it can be expanded in a double Fourier series. Expressing this series in terms of $g(u,\theta)$ rather than $h(\vartheta,\gamma)$ and comparing it with the expression for $\widehat{f}(k,s)$ in (22), we obtain

$$\widehat{f}(2\pi m/T, nL) = \int_0^T \int_0^{2\pi/L} g(u,\theta)e^{-i(2\pi mu/T+n\theta L)}d\theta du \tag{23}$$

and

$$g(u,\theta) = \frac{L}{2\pi T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\widehat{f}(2\pi m/T, nL)e^{i(2\pi mu/T+n\theta L)}. \tag{24}$$

Finally, introducing $g(u,\theta) = f(e^{u+i\theta})e^u$ and

$$e^{-i(2\pi mu/T+n\theta L)} = r^{-i2\pi m/T}e^{-inL\theta} = (z/|z|)^{-nL}\,|z|^{-i2\pi m/T},$$

into (23) and (24), we easily arrive at

$$\widehat{f}(2\pi m/T, nL) = \frac{i}{2}\int f(z)\left(\frac{z}{|z|}\right)^{-nL}|z|^{-i2\pi m/T-1}dzd\overline{z}$$

and

$$f(z) = \frac{L}{2\pi T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\widehat{f}(2\pi m/T, nL)\left(\frac{z}{|z|}\right)^{-nL}|z|^{-i2\pi m/T-1}, \tag{25}$$

which we call the *projective Fourier series* of the function f satisfying (21).

6 The discrete projective Fourier transform

Now we will proceed with our main approximation procedure. We approximate the integral in (23) by a double Riemann sum

$$\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\frac{2\pi T}{LNM}g(kT/M, l2\pi/LN)e^{-2\pi i(mk/M+nl/N)} \tag{26}$$

and define $f_{k,l}$ and $\widehat{f}_{m,n}$ as follows

$$f_{k,l}e^{kT/M} = \frac{2\pi T}{LNM}g(kT/M, l2\pi/LN) \tag{27}$$

and

$$\widehat{f}_{m,n} = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1}f_{k,l}e^{kT/M}e^{-2\pi i(km/M+ln/N)}. \tag{28}$$

Clearly, the sequence $\{\widehat{f}_{m,n}\}$ is doubly periodic with period (M, N):

$$\widehat{f}_{m+M,n+N} = \widehat{f}_{m,n}.$$

Now, introducing these definitions and the relation

$$z_{m,n} = e^{mT/M}e^{i\frac{2\pi n}{LN}} = r_m e^{i\theta n}$$

into (26) and (28) we arrive at

$$\widehat{f}_{m,n} = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1}f_{k,l}\left(\frac{z_{k,l}}{|z_{k,l}|}\right)^{-nL}|z_{k,l}|^{-i2\pi m/T-1} \tag{29}$$

and

$$f_{k,l} = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\widehat{f}_{m,n}\left(\frac{z_{k,l}}{|z_{k,l}|}\right)^{nL}|z_{k,l}|^{i2\pi m/T-1} \tag{30}$$

for functions $f_{k,l}$ satisfying

$$f_{k+mM,l+nL} = f_{k,l}e^{-mT}$$

where m, $n \in \mathbb{Z}$.

The expressions in (29) and (30) are called the (M, N)-*point discrete projective Fourier transform* and its *inverse discrete projective Fourier transform*, respectively, and (29) provides an approximation

$$\widehat{f}_{m,n} \approx \widehat{f}(2\pi m/T, nL)$$

of the projective Fourier transform (16) of the pattern.

The inverse transform in (30) demonstrates the projectively adapted properties as for any $g \in \mathbf{SL}(2,\mathbb{C})$

$$f(g^{-1} \cdot z_{k,l}) = \frac{L}{2\pi T} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \widehat{f}_{m,n} \left(\frac{g^{-1} \cdot z_{k,l}}{|g^{-1} \cdot z_{k,l}|} \right)^{nL} |g^{-1} \cdot z_{k,l}|^{\frac{i2\pi m}{T} - 1} \tag{31}$$

produces the corresponding projective distortion of the pattern

$$f(z_{k,l}) = \frac{LMN}{2\pi T} f_{k,l},$$

and this distortion is expressed only in terms of the projective discrete Fourier transform $\widehat{f}_{m,n}$ given in (29).

7 FFT in reconstructions of projectively distorted patterns

In conclusion, the discrete projective Fourier transform (29) and its inverse (30) demonstrate that we need the same number of operations (an operation is defined to mean a complex multiplication followed by a complex addition) as in the classical case, that is, at least $(M-1)^2(N-1)^2$ operations for $M \times N$ sampled points.

Thus, to compute the projective Fourier transform shouldn't be more difficult than in the case of the classical Fourier transform, and in fact, one can adapt in rather straightforward way the algorithms of FFT as they are presently used in image processing.

Moreover, (31) shows that one can adapt rather easily the FFT algorithms for a fast reconstruction of any projective distortion of a pattern. These projectively adapted algorithms are presented in[4] , where also numerical examples are given.

8 REFERENCES

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