Harmonic Analysis on SL(2,$\mathbb{C}$) and Projectively Adapted Pattern Representation

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Abstract

Among all image transforms the classical (Euclidean) Fourier transform has had the widest range of applications in image processing. Here its projective analogue, given by the double cover group $\mathbf{SL}(2,\mathbb{C})$ of the projective group $\mathbf{PSL}(2,\mathbb{C})$ for patterns, is developed. First, a projectively invariant classification of patterns is constructed in terms of orbits of the group $\mathbf{PSL}(2,\mathbb{C})$ acting on the image plane (with complex coordinates) by linear-fractional transformations. Then, $\mathbf{SL}(2,\mathbb{C})$-harmonic analysis, in the noncompact picture of induced representations, is used to decompose patterns into the components invariant under irreducible representations of the principal series of $\mathbf{SL}(2,\mathbb{C})$. Usefulness in digital image processing problems is studied by providing a camera model in which the action of $\mathbf{SL}(2,\mathbb{C})$ on the complex image plane corresponds to, and exhaust, planar central projections as produced when aerial images of the same scene are taken from different vantage points. The projectively adapted properties of the $\mathbf{SL}(2,\mathbb{C})$-harmonic analysis, as applied to the problems in image processing, are confirmed by computational tests. Therefore, it should be an important step in developing a system for automated perspective-independent object recognition.

Keywords: Fourier analysis, irreducible representations of the group $\mathbf{SL}(2,\mathbb{C})$, harmonic analysis on $\mathbf{SL}(2,\mathbb{C})$, projectively invariant classification of patterns, projectively adapted pattern representation.

Mathematics subject classification: 43A30, 43A65, 43A85, 68T10, 68U10.

1 Introduction

Applications of noncommutative harmonic analysis have expanded at much slower rate that those of the classical Fourier analysis on $\mathbb{R}^n$, especially after the fast algorithms for computing Fourier transforms have been developed. Noncommutative harmonic analysis on the two-dimensional sphere S^2, the symmetric space of the rotation group $\mathbf{SO}(3)$, and non-Euclidean analogues of Fourier series and integrals with many applications, are discussed in [21]. The problems of efficient computation of spherical harmonics, i.e., Fourier transforms of functions on S^2, with emphasis on applications in computer vision, have been recently studied in [8]. Fourier transforms on discrete noncommutative groups with efficient algorithms and suggestions for interesting applications in statistics have been attacked in [6], [7] and [5].

In this article we develop a projective analogue of Fourier analysis and show how it provides projectively adapted pattern representation by decomposing patterns in terms of their projective Fourier transforms.

The standard Fourier analysis provides important tools in signal and image processing, used for example, in telecommunication (telephone and television), in transmission and analysis of satellite images, and in medical imaging (echography, tomography, and nuclear magnetic resonance). Recently, a question of deficiency of the Fourier decomposition of aerial images has been raised in [1].

The authors in [1] noted that the Fourier transform of the distorted image under a linear transformation can be computed from the Fourier transform of the original image by a simple rule, see [9]. However, when a central projection or a Möbius transformation is applied, the relationship between Fourier transforms of the original image and a distorted image is no longer feasible.

They pointed out that a representation of images such that one has a closed-form relation between the representations of the original image and its projective distortions is an important step in developing a system for automated perspective-independent object recognition. Moreover, they constructed a variant of the Fourier transform by projecting Fourier basis functions from the surface of a sphere onto an image plane, and then decomposing an image with respect to this family of projected functions.

Motivated by this work, we show by constructing a camera model that the orbit of a pattern under the group $\mathbf{PSL}(2,\mathbb{C})$ acting by linear-fractional transformations on the (complex) image plane, contains all generic projective distortions of the pattern. We use "generic" to indicate that we have to exclude some projections, for example, when the images of a pattern are one-dimensional. They form a lower dimensional subset in the space of all projective transformations between images.

Further, we develop a projective analogue of Fourier analysis. It is a noncommutative, $\mathbf{SL}(2,\mathbb{C})$-harmonic analysis on the complex line which decomposes L^2-space of functions (patterns' pixel gray-level functions) into irreducible invariant subspaces, that is, subspaces transforming under the irreducible unitary representations of the principal series of $\mathbf{SL}(2,\mathbb{C})$. The main results of the analysis are the projective analogue of the (Euclidean) Fourier decomposition, given here in terms of characters of the Borel subgroup of $\mathbf{SL}(2,\mathbb{C})$ with the coefficients of the decomposition given by the projective Fourier transform, and the corresponding Plancherel's theorem. Since the group $\mathbf{SL}(2,\mathbb{C})$ is the double cover of $\mathbf{PSL}(2,\mathbb{C})$, this decomposition should provide an important tool in a projectively adapted image processing. This point is further discussed and confirmed by computer simulations at the end of this article.

It seems that the projective harmonic analysis as developed in this work has not been presented elsewhere, though, we use a well known methods in its construction.

As it has been said (see the comments for Section 7 given on p. 546 in [11]), there is no royal road through contemporary semisimple harmonic analysis, including infinite-dimensional representation theory, since most treatments and surveys are highly abstract and technical. However, we refer to a constructive approach given in [10], and to an overview based on examples in [13], as well as [22]. Nontechnical expositions with a historical perspective can be found in [11] and [15].

The choice of presentation of the subject is made here to bridge as much as possible the gap between abstract theories of group representation and noncommutative harmonic analysis, and the computer vision and pattern analysis communities. In particular, we will stress in the construction given here the relation between the projective Fourier transform and the classical (Euclidean) Fourier transform.

2 Transformation groups in machine vision

2.1 Projective geometry in vision

Euclidean group acting on the points of space consists of the group of rigid motions of objects in space. An image of an object is its planar central projection, i.e., a projection on some image plane along rays intersecting at a given point (called the center of a projection) that is not on the plane. These projections are called perspective projections. Intuitively, the set of all perspective projections of a three-dimensional space consists of a two-dimensional projective space. Any two such images are related by a projective transformation, also called a homography.

The definition of a finite dimensional projective space in this context is the following.

Definition 1 *Let V be a vector space over the field K. The projective space derived from V, denoted by $P(V)$, is the quotient of $V \setminus 0$ by the equivalence relation "$x \sim y$ if and only if $y = \lambda x$ for some $\lambda \in K$". The dimension of $P(V)$ is $dim V - 1$. The canonical projection is $p : V \backslash 0 \to P(V)$.*

A projective space is called real if $K = \mathbb{R}$, and complex if $K = \mathbb{C}$.

Example 2 *For every integer $n \geq 1$, $P^n(K) = P(K^{n+1})$ is called the standard projective space of dimension n over the field K.*

If we take a basis $\{e_i\}_{i=1,..,n+1}$ of V, then every $x \in V$ has the representation $(x_1, .., x_{n+1})$ with respect to this basis. Moreover, every point $y \in P(V)$ is of the form $y = p(x_1, .., x_{n+1})$ where p is the canonical projection. The homogeneous coordinates of the point $y \in P(V)$ (with respect to the basis $\{e_i\}_{i=1,..,n+1}$) is any set $(\lambda x_1, .., \lambda x_{n+1})$ with some $\lambda \in K_* = K \backslash 0$ and fixed $x_1, .., x_{n+1}$.

Definition 3 *Let $P(V)$ and $P(V')$ be projective spaces. A projective transformation, or a homography is the map $g : P(V) \to P(V')$ satisfying $g \circ p = p \circ f$ where $f : V \to V'$ is an isomorphism of vector spaces.*

Projective transformations $P(V) \to P(V)$ form a group under composition of maps, called the *projective group* and denoted by $\mathbf{PSL}(V)$. There is a group isomorphism (see, [3])

$$\mathbf{PSL}(V) \cong \mathbf{GL}(V)/K_* Id. \tag{1}$$

We give now the explicite calculation of (1) for the complex projective line $P^1(\mathbb{C})$ and latter we will demonstrate in Theorem 7 and Corollary 8 the relevence of $P^1(\mathbb{C})$ to machine vision problems.

The equivalence relation "$x \sim y$ if and only if $y = \lambda x$ for some $\lambda \in \mathbb{C}$" in Definition 1 in the case of the projective space $P^n(\mathbb{C})$, implies that the points of the projective line $P^1(\mathbb{C}) = P(\mathbb{C}^2)$ can be identified with the slopes of the lines $z_2 = \mu z_1$ in $\mathbb{C}^2$ if we add one extra 'slope', denoted by ∞, for the line $z_1 = 0$. The point ∞ is called the point at infinity. Note that the set of lines $z_2 = \mu z_1$ (i.e., slopes) forms an open affine patch $\mathbb{C}$ in $P^1(\mathbb{C})$. The procedure of adding the point at infinity is referred to as the projective completion of an affine space.

However, the definition of a projective space given above treats all points of $P^1(\mathbb{C})$ on equal footing. Therefore, if one removes from $\mathbb{C}^2$ any line l passing through the origin, one will get a natural affine structure on $P(\mathbb{C}^2)\backslash P(l)$.

Now blending the idea of homogeneous coordinates with the idea of constructing affine patches, we take the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ (i.e., $e_1 = (1, 0)$ and $e_2 = (0, 1)$) and define the line $l_i = z_i^{-1}(0)$. The points of $H_i = P(\mathbb{C}^2)\backslash P(l_i)$ are those whose homogeneous coordinates (z_1, z_2) satisfy $z_i \neq 0$. Now, H_i is isomorphic to the hyperplane $z_i^{-1}(1) = H_i + e_i$ which has an affine frame $f_i = \{e_i\} \cup \{e_i + e_j\}_{j \neq i}$. Then, in the affine frames f_i $(i = 1, 2)$ the charts $P(\mathbb{C}^2)\backslash P(l_i)$ are given by

$$\pi_1 : P(\mathbb{C}^2)\backslash P(l_1) \ni p(z_1, z_2) \longmapsto \frac{z_2}{z_1} \in \mathbb{C}$$

and

$$\pi_2 : P(\mathbb{C}^2)\backslash P(l_2) \ni p(z_1, z_2) \longmapsto \frac{z_1}{z_2} \in \mathbb{C}$$

respectively.

Thus, the complex projective space $P^1(\mathbb{C})$ can be regarded as the Riemann S^2 sphere which is obtained by gluing together two copies of $\mathbb{C}$ by the map $\xi \longmapsto 1/\xi$ on $\mathbb{C}_*$.

The complex projective space $P^1(\mathbb{C})$ can be also regarded as the one-point compactification of $\mathbb{C}$, denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In fact, the stereographic projection from, for example, the north pole of the sphere S^2, with the north pole projected on ∞ shows that $P^1(\mathbb{C}) = \widehat{\mathbb{C}}$.

We conclude this discussion by expressing the projective transformations $g : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ in the local chart π_1 we have introduced above. Using (1), we can write the matrix of g in the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ as

$$M(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where complex numbers a, b, c, and d satisfy $\det M(g) = ad - cb \neq 0$. Then, a simple calculation gives

$$\pi_1 \circ g \circ \pi_1^{-1} : \xi \longmapsto \frac{d\xi + c}{b\xi + a}. \tag{2}$$

In conclusion, we can assume that $M(g) \in \mathbf{PSL}(\mathbb{C}^2)$ since it follows from (2) that both $\frac{\pm 1}{det M(g)} M(g)$ and $M(g)$ give the same orbit of $\xi \in \mathbb{C}$.

2.2 The Euclidean group acting on $\mathbb{R}^3$

We start with Euclidean group $\mathbf{E} = \mathbf{SO}(3) \dot{\times} \mathbb{R}^3$. It is a semidirect product of the special orthogonal group $\mathbf{SO}(3)$ (the group of all 3×3 real matrices with determinant 1), and the additive group $\mathbb{R}^3$. The group operation is the following

$$(R_1, b_1)(R_2, b_2) = (R_1 R_2, R_1 b_2 + b_1).$$

Then, $\mathbb{R}^3$ is a normal subgroup with $\mathbf{SO}(3)$ as quotient group.

The action of $\mathbf{E}$ on $\mathbb{R}^3$ is given by

$$(R, b)\, u = Ru + b \tag{3}$$

where $(R, b) \in \mathbf{E}$, $u \in \mathbb{R}^3$ and the matrix R is acting on column vectors

$$u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

Thus, (R, b) in (3) first rotates the vector u giving Ru, and then translates Ru by b.

In order to specify rotations quantitatively, we introduce Euler angles. We assume the following parametrization of the rotation matrix R

$$\begin{aligned} R &= R(\psi_1, \phi, \psi_2) \\ &= R(\psi_1)R(\phi)R(\psi_2) \end{aligned} \tag{4}$$

where $R(\psi_i)$ $(i = 1, 2)$ is a matrix of the form:

$$R(\psi_i) = \begin{pmatrix} \cos\psi_i & 0 & \sin\psi_i \\ 0 & 1 & 0 \\ -\sin\psi_i & 0 & \cos\psi_i \end{pmatrix}$$

and $R(\phi)$ has the form:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here ψ_1 is the first rotation angle about the x_2-axis, ϕ is the following rotation angle about the rotated x_3-axis, and finally, ψ_2 is the last rotation angle about the x_2-axis rotated by the previous two angles. Note that because of historical reasons, Euler angles refer to rotations about the axes of the moved system and therefore appear in reverse order.

2.3 The group $\mathbf{SL}(2,\mathbb{C})$ acting on $\widehat{\mathbb{C}}$

The group $\mathbf{SL}(2,\mathbb{C})$, consisting of all 2×2 complex matrices of determinant 1, acts on nonzero column vectors

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2.$$

If we take a local coordinate system on $\widehat{\mathbb{C}}$ given by $z_1 = 1$ (any point $\xi \neq \infty$ of $\widehat{\mathbb{C}}$ is then identified with the point where the line $z_2 = \xi z_1$ intersects the line $z_1 = 1$), one can easily check that the action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C})$$

on $\mathbb{C}^2$ induces the following action on slopes of the lines $z_2 = \mu z_1$,

$$\mu \longmapsto \mu' = \frac{d\mu + c}{b\mu + a}.$$

and consequently, $\mathbf{SL}(2, \mathbb{C})$ acts on $\widehat{\mathbb{C}}$ by linear-fractional transformations given by

$$\mathbf{SL}(2, \mathbb{C}) \ni g : \xi \longmapsto g \cdot \xi = \frac{d\xi + c}{b\xi + a} \tag{5}$$

if $\xi \neq \frac{-a}{b}$. Moreover, $g \cdot \frac{-a}{b} = \infty$, $g \cdot \infty = \frac{d}{b}$, and if $b = 0$, $g \cdot \infty = \infty$. Note that (5) agrees with (2).

However, note that both

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \tag{6}$$

give the same orbit of ξ. Therefore, as we have discussed it in Section 2.1, the projective group for patterns is the group

$$\mathbf{PSL}(2, \mathbb{C}) = \mathbf{SL}(2, \mathbb{C})/\{\pm Id\}$$

where Id is the identity matrix. It is enough to take the quotient of $\mathbf{SL}(2, \mathbb{C})$ by $\pm Id$ because of the normalization condition, namely, that every matrix in $\mathbf{SL}(2, \mathbb{C})$ has a determinant of 1. Now, since $\{\pm Id\}$ is discrete, the quotient map of $\mathbf{SL}(2, \mathbb{C})$ onto $\mathbf{SL}(2, \mathbb{C})/\{\pm Id\}$ is a covering map. Moreover, $\mathbf{SL}(2, \mathbb{C})$ is simply connected (because $\mathbf{SU}(2) \subset \mathbf{SL}(2, \mathbb{C})$ is simply connected being diffeomorphic with $\mathbf{SO}(3)$, see the proof of the next lemma) and therefore it is the universal double cover of $\mathbf{PSL}(2, \mathbb{C})$.

Lemma 4 *The maximal compact subgroup of* $\mathbf{SL}(2,\mathbb{C})$*,*

$$\mathbf{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1 \right\},$$

is the universal double cover of the group of rotations $\mathbf{SO}(3)$*.*

Proof. *This lemma is proved here using the explicit relation between* a *and* b *and Euler angles* ψ_1*,* ϕ *and* ψ_2 *centered at* $(0, 1, 0)$ *since the pinhole of the camera will be located at the origin and the image plane will be given by* $x_2 =$

1. *This relation gives a two-to-one diffeomorphism between* $\mathbf{SU}(2)$ *and* $\mathbf{SO}(3)$ *(*$\mathbf{SO}(3)$ *acts here on the unit sphere* $\mathbf{S}^2$ *that is centered at* $(0,1,0)$*) and defines a double covering map* $\pi : \mathbf{SU}(2) \to \mathbf{SO}(3)$*. By this diffeomorphism,* $\mathbf{SU}(2)$ *is simply connected and therefore* $\mathbf{SU}(2)$ *is the universal double covering group of* $\mathbf{SO}(3)$.

To this end, in order to express $k \in \mathbf{SU}(2)$ *as a function* $k(\psi_1,\ \phi,\ \psi_2)$ *of the Euler angles we set*

$$k = \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix}$$

and we define the traceless hermitian matrix $\Omega(u)$ *for each* $u = (x_1, x_2 - 1, x_3)^t$ *by*

$$\Omega(u) = \begin{pmatrix} -x_2 + 1 & x_3 - ix_1 \\ x_3 + ix_1 & x_2 - 1 \end{pmatrix}.$$

Next, we prove that the matrix R *defined by*

$$u' = Ru \tag{7}$$

where $u' = (x_1', x_2' - 1, x_3')^t$*, if and only if*

$$\Omega(u') = k\Omega(u)k^{-1} \tag{8}$$

is a rotation matrix. In fact, for every $k \in \mathbf{SU}(2)$*,* $k\Omega(u)k^{-1}$*is a traceless hermitian matrix, which shows that* R *exists. Moreover, since*

$$\begin{aligned} -\|u\| &= \det \Omega(u) \\ &= \det \Omega(u') = -\|u'\|, \end{aligned}$$

R *is a rotation centered at* $(0,1,0)$.

Finally, if R *is parametrized by Euler angles* ψ_1*,* ϕ *and* ψ_2*, then solving the set of equations in (7) and (8) will yield the diffeomorphism* $\mathbf{SO}(3) \ni (\psi_1,\ \phi,\ \psi_2) \to k(\psi_1,\ \phi,\ \psi_2) \in \mathbf{SU}(2)$ *given explicitly by (see also [4], p. 188)*

$$a = \pm \cos\left(\frac{\phi}{2}\right) e^{i(\psi_1+\psi_2)/2} \tag{9}$$

and

$$b = \pm i \sin\left(\frac{\phi}{2}\right) e^{i(\psi_1-\psi_2)/2} \tag{10}$$

(both with " + " or both with " − "). One checks that the point $(\psi_1', \phi', \psi_2')$ *satisfying* $k(\psi_1',\ \phi',\ \psi_2') = -k(\psi_1,\ \phi,\ \psi_2)$ *is not in the same chart on* $\mathbf{SO}(3)$ *as* (ψ_1, ϕ, ψ_2). ■

The more detailed discussion of the group manifolds of $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ may be found in [4]. Also, for the background material on covering spaces we refer to [19].

3 The $\mathbf{PSL}(2, \mathbb{C})$-camera for patterns

We start with the description of a pinhole camera. The pinhole, or optical center, of the camera is located at the center of the planar projection. This is the point where the incoming rays of light intersect each other, giving an image on the image plane. The ray perpendicular to the image plane is usually assumed to be the optical axis and points in the viewing direction. The point where the optical axis intersects the image plane is called the principal point.

In order to formulate a camera model quantitatively, we consider an image plane to be the plane $x_2 = 1$ in $\mathbb{R}^3 = \{(x_1, x_2, x_3)^t : x_i \in \mathbb{R}\}$. Then, the image of $(x_1, x_2, x_3)^t$ on the image plane is given by the projection $j : \mathbb{R}^3 \to \mathbb{C}$,

$$j\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{x_3 + ix_1}{x_2} \tag{11}$$

where $\mathbb{C}$ is identified with the plane $x_2 = 1$ in $\mathbb{R}^3$ by using complex coordinates $x_3 + ix_1$.

This image plane can be regarded as the extended complex line $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with its affine piece $\mathbb{C}$ given by the equation $x_2 = 1$ if we require that

$$j\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = \infty.$$

In order to introduce the action of the group $\mathbf{SL}(2,\mathbb{C})$ on the complex image plane we should take

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1 = x_2 + iy,\ z_2 = x_3 + ix_1 \right\}.$$

Indeed, if we choose a local coordinate system on $\widehat{\mathbb{C}}$ given by $z_1 = 1$, then the points ξ of the affine patch $\mathbb{C}$ are identified with the points $(1, x_3 + ix_1)$ (i.e., where the lines $z_2 = \xi z_1$ intersect the line $z_1 = 1$). Note that the choice of the image plane will result in the agreement with the left action used in this article in the construction of the principal series representations of $\mathbf{SL}(2,\mathbb{C})$.

Consequently, the action of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \mathbf{SL}(2,\mathbb{C})$$

on the image plane $\widehat{\mathbb{C}}$ is given in (5), i.e.,

$$g \cdot \xi = \frac{d\xi + c}{b\xi + a}. \tag{12}$$

Assuming that patterns *"live"* on the image plane $x_2 = 1$, the projective distortions of a pattern are generated by translating or rotating the pattern to form its *"virtual"* space position and then projecting by (11) on the image plane, and by all finite iterations of these basic distortions. In fact, patterns are defined on $\widehat{\mathbb{C}}$, however, we work with them in the affine patch given by $z_1 = 1$. Thus, we state the following

Definition 5 *A* pattern *P is a function f, its intensity function, with a compact support D in the image plane $\widehat{\mathbb{C}}$. We write*

$$P = \{f : D \to \mathbb{R}\}. \tag{13}$$

Its projective distortion is a pattern P_g where

$$P_g = \{fg^{-1} : gD \to \mathbb{R}\}, \quad g \in G^{\times} \tag{14}$$

where $G^{\times}$ is the set generated by all finite iterations of distortions produced by rotations or translations as described before.

Note that P_g "inherits" the intensity of P at corresponding points, i.e., $f'(\xi') = f(\xi)$ where $\xi' = g\xi$ and $f' = fg^{-1}$. It is a useful starting point of the image analysis, which may be realized by *ex posto facto calibration*, see [1]. We refer to Definition 5 as to a camera model for patterns.

In order to describe $G^{\times}$ in (14), we start with some preliminary facts.

It follows from Lemma 4 that for a given $r = R(\psi_2, \phi, \psi_1) \in \mathbf{SO}(3)$ there is the element $k \in \mathbf{SU}(2) \subset \mathbf{SL}(2, \mathbb{C})$ acting by

$$\begin{aligned} k \cdot \xi &= \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix} \xi \\ &= \frac{a\xi + b}{-\overline{b}\xi + \overline{a}} \end{aligned} \tag{15}$$

such that $jr = kj$, where a and b given in (9) and (10), respectively and j is given in (11).

The corresponding result for the translational part of Euclidean group is treated in the following lemma.

Lemma 6 *Given $b = (b_1, b_{2,} b_3)^t \in \mathbb{R}^3$. There is the map $h : \mathbb{C} \to \mathbb{C}$ given by*

$$h \cdot \xi = \frac{\alpha^{-1}\xi + \beta}{\alpha} \tag{16}$$

such that $jb = hj$, $\alpha \neq 0$ is given by

$$\begin{aligned} \alpha &= (1 + b_2)^{1/2} \quad \textit{if} \quad 1 + b_2 > 0 \\ \alpha &= i\,(|1 + b_2|)^{1/2} \quad \textit{if} \quad 1 + b_2 < 0 \end{aligned}$$

and

$$\beta = (b_3 + ib_1)\alpha^{-1}. \tag{17}$$

Proof. *If $(x_1, 1, x_3)^t$ is translated by $b = (b_1, b_{2,} b_3)^t$ and then projected by (11), we obtain $(x_1', 1, x_3')^t$ where*

$$x_3' + ix_1' = \frac{x_3 + ix_1 + b_3 + ib_1}{1 + b_2} = \frac{\alpha^{-1}\xi + \beta}{\alpha}$$

with α given in (17) and β in (17). ■

The conditions in (17) exclude the translation vectors $b = (b_1, -1_, b_3)^t$. All points $(x_1, 1, x_3)^t$ when translated by these vectors have the second coordinate zero and therefore are projected by (11) on ∞. This exclusion agrees with the idea, reappearing frequently in this work, of admitting generic projections.

A simple implication of the last lemma is the factorization of h. We have two cases:

(1) If $1 + b_2 > 0$, then

$$h = a\overline{n} \tag{18}$$

(2) If $1 + b_2 < 0$, then

$$h = \varepsilon a\overline{n} \tag{19}$$

where

$$\varepsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and

$$a \in \mathbf{A} = \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} : \delta \in \mathbb{R}_+ \right\}, \tag{20}$$

$$\overline{n} \in \overline{\mathbf{N}} = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \mathbb{C} \right\}. \tag{21}$$

In the remaining part of this section, we prove two main results (Theorem 7 and Corollary 8) that conclude the camera model.

Theorem 7 *The set $G^\times$ in (14) is the projective group*

$$\mathbf{PSL}(2, \mathbb{C}) = \mathbf{SL}(2, \mathbb{C})/\{\pm Id\}.$$

Proof. *In order to prove this theorem first we show that for a given pattern $P = \{f : D \to R\}$ and s in $\mathbf{SO}(3)$, or in $\mathbb{R}^3$, there is the corresponding s' such that $js = g$ and $js' = g^{-1}$, that is, $(P_g)_{g^{-1}} = P$. The existence of such s' follows from (15) and (16). Now, using the decomposition (for example, see [13], p. 126),*

$$\mathbf{SL}(2, \mathbb{C}) = \mathbf{SU}(2)\mathbf{A}\mathbf{SU}(2)$$

where $\mathbf{A}$ is defined in (20), we conclude that $G^\times = \mathbf{PSL}(2, \mathbb{C})$. $\blacksquare$.

To conclude the camera model we must prove that the action of the group $\mathbf{PSL}(2, \mathbb{C})$ on the image plane covers all projective transformations between different planes. This fact is a corollary of Theorem 7 and the explicit calculation of the projective group in affine charts of the projective line.

To this end, recall that the projective transformations $P(E) \to P(E)$ form the *projective group* of E under composition of maps, denoted by $\mathbf{PSL}(E)$ and isomorphic with $\mathbf{GL}(E)/K_* Id$. In the case of $E = \mathbb{C}^2$ (i.e., $P(E) = P^1(\mathbb{C})$), the group $\mathbf{PSL}(\mathbb{C}^2)$ can be identified with the group $\mathbf{PSL}(2, \mathbb{C})$ acting on $P^1(\mathbb{C}) \cong \widehat{\mathbb{C}}$ by linear-fractional transformations. An explicit calculation of this fact has been given in (2).

Corollary 8 *In the camera model,* $\mathbf{PSL}(2, \mathbb{C})$ *acting on the image plane* $x_2 = 1$ *identified with* $\mathbb{C}$ *models all generic projective transformations of a given pattern.* ■

We call the camera model for patterns the $\mathbf{PSL}(2, \mathbb{C})$-camera.

4 The relation of the $\mathbf{PSL}(2, \mathbb{C})$-camera with other camera models

The camera models are used in machine vision research to extract information about a scene from its images, and for a dynamic scene, also predict the future locations of objects in the scene. They can be classified into two categories; calibrated and uncalibrated with the different types of cameras in each of the categories.

The projective camera is the most general camera. It projects the points of space on an image plane. This projection can be written as 3×4 real matrix in the homogeneous coordinates in space (the world coordinates) and in an image plane (the camera coordinates). Since scale is arbitrary for homogeneous coordinates and the mapping places no restrictions on the coordinates, it is called an uncalibrated camera. The other less general uncalibrated camera is the affine camera. It corresponds to a projective camera with the center of projection on the plane at infinity; which means that all projecting rays are parallel. This camera results in the composed effect of affine transformations between the world and camera coordinates, parallel projections onto the image plane and affine transformation of the image plane coordinates.

All other cameras, including calibrated cameras, are obtained from the two uncalibrated cameras by specifying coordinate frames, transformations between frames, and camera parameters (i.e., "calibration") such as the focal length or the principal point.

The most general calibrated camera is the perspective camera. This model employs central (i.e., perspective) projections, reducing to the previously described pinhole camera when the world and camera coordinate frames are related by an Euclidean transformation. Further restrictions results in less general camera models, such as the weak perspective camera (when the depth variation of objects along the viewing line is small compared with the viewing distance) or the orthographic camera (orthographic projections).

Any of the discussed above cameras is defined by projections characterized by the corresponding 3×4 homogeneous matrix, see [16] , where also a nice discussion of the cameras with emphasis on both machine vision and projective geometry is given. The classification of cameras with the corresponding matrices is given in [18] p. 42.

A set of 3×4 matrices that characterize a corresponding camera does not form a group. In contrast, the $\mathbf{PSL}(2,\mathbb{C})$-camera model constructed in Section 3 is a camera that is characterized by the projective transformations between the image planes of two different perspective projections. These projective transformations are given in terms of $\mathbf{PSL}(2,\mathbb{C})$ acting on the camera image plane by linear-fractional transformations and they form the *projective group* under composition (with the point ∞ included).

5 Irreducible unitary representations of $\mathbf{SL}(2,\mathbb{C})$

It follows from the $\mathbf{PSL}(2,\mathbb{C})$-camera model that the projectively adapted harmonic analysis for patterns could be formulated on the homogeneous space $\widehat{\mathbb{C}}$ of the group $\mathbf{SL}(2,\mathbb{C})$. This analysis of patterns' intensity functions is given in terms of the irreducible unitary representations of $\mathbf{SL}(2,\mathbb{C})$. There are different realizations of the irreducible unitary representations of $\mathbf{SL}(2,\mathbb{C})$. The realization which is referred to as induced representations of $\mathbf{SL}(2,\mathbb{C})$ in *noncompact picture,* see [13], p. 169*,* is the most convenient to work with analytically and will be used in this work. Before we write down the irreducible unitary representations of $\mathbf{SL}(2,\mathbb{C})$, first, for completeness, we state general definitions and discuss how irreducible unitary representations of $\mathbf{SL}(2,\mathbb{C})$ can be realized as induced representations.

5.1 Representations

Let $\mathbf{G}$ be a locally compact group, then

Definition 9 *A representation of* $\mathbf{G}$ *on a complex Hilbert space* V *is a homomorphism* $\mathcal{R}$ *of* $\mathbf{G}$ *into the group of bounded linear operators on* V *with bounded inverses such that the resulting map of* $\mathbf{G}\times V$ *into* V *is continuous.*

An *invariant subspace* of $\mathcal{R}$ is a vector subspace $U \subset V$ such that $\mathcal{R}(g)U \subset U$ for all $g \in \mathbf{G}$.

Definition 10 *A representation is irreducible if it has no closed invariant subspaces other then* 0 *and* V*.*

Such a representation is *unitary* if $\mathcal{R}(g)$ is unitary and two (unitary) representations of $\mathbf{G}$, $\mathcal{R}$ on V and $\mathcal{R}'$ on V' are *(unitarily) equivalent* if there is a (unitary) bounded linear $T: V \to V'$ with a bounded inverse such that $\mathcal{R}'(g)T = T\mathcal{R}(g)$ for all $g \in \mathbf{G}$.

Example 11 *(1)* $\mathbf{G} = \mathbf{SL}(2,\mathbb{C})$, $V = L^2(\mathbb{C}^2)$ *(here* $\mathbb{C}^2 = \mathbb{R}^4$*) and* $\mathcal{R}(g)f(z) = f(g^{-1}z)$ *with the usual inner product on* V *is a unitary representation.*

(2) $\mathbf{G}$ *is a Lie group,* $V = L^2(\mathbf{G}, d_l x)$ *taken with respect to a left-invariant measure* $d_l x$ *and* $\mathcal{R}(g)f(x) = f(g^{-1}x)$ *is the left regular representation of* $\mathbf{G}$*. The right regular representation* $\mathcal{R}'$ *is given by* $\mathcal{R}'(g)f(x) = f(xg)$ *on* $L^2(\mathbf{G}, d_r x)$ *where* $d_r x$ *is a right-invariant measure on* $\mathbf{G}$*.*

5.2 Induced representations of $\mathbf{SL}(2,\mathbb{C})$

We start by showing that, in the sense discussed below,

$$\mathbf{SL}(2,\mathbb{C})/\mathbf{B} \approx \mathbb{C} = \mathbb{R}^2$$

where $\mathbf{B}$ is the Borel (”parabolic”) subgroup of $\mathbf{SL}(2,\mathbb{C})$,

$$\mathbf{B} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}_*, \beta \in \mathbb{C} \right\} \tag{22}$$

with $\mathbb{C}_* = \mathbb{C}\backslash\{0\}$ being the multiplicative group of complex numbers. Note that $\mathbf{B} = \mathbf{MAN}$ with $\mathbf{M}$ being the maximal torus in $\mathbf{SU}(2)$ (generated by rotations in the image plane),

$$\mathbf{M} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\},$$

$\mathbf{A}$ defined in (20) and $\mathbf{N}$ containing all the upper triangular matrices with one along diagonals.

Indeed, the Gauss decomposition

$$\mathbf{SL}(2,\mathbb{C}) \doteq \overline{\mathbf{N}}\mathbf{MAN} = \overline{\mathbf{N}}\mathbf{B}, \tag{23}$$

where "$\doteq$" means that equality holds except some lower dimensional subset, i.e., almost everywhere, is given by

$$g = \overline{n}(g)m(g)a(g)n(g) \tag{24}$$

with

$$\overline{n}(g) = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix}; \qquad m(g) = \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{|\alpha|}{\alpha} \end{pmatrix};$$

$$a(g) = \begin{pmatrix} |\alpha| & 0 \\ 0 & |\alpha|^{-1} \end{pmatrix}; \qquad n(g) = \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}$$

where

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is such that $\alpha \neq 0$.

Therefore,

$$\mathbf{SL}(2,\mathbb{C})/\mathbf{B} \doteq \overline{\mathbf{N}} \cong \mathbb{C}$$

because a Lie group $\overline{\mathbf{N}} = \{\overline{n}(g) : g \in \mathbf{SL}(2,\mathbb{C})\}$ is Lie isomorphic with the additive group of complex numbers $\mathbb{C}$ through the action

$$\overline{n}(g)\xi = \xi + \frac{\gamma}{\alpha}.$$

In most of the contemporary research papers and monographs, the irreducible unitary representations on a semisimple groups are constructed as the induced representations. In the case of the group $\mathbf{SL}(2,\mathbb{C})$, the irreducible unitary representation are induced from the one dimensional unitary representations of $\mathbf{M}$ given by

$$\begin{aligned}\pi_k(m) &= \pi_k\left(e^{i\theta}\right)\\ &= e^{ik\theta}\end{aligned}$$

where

$$m = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathbf{M}.$$

This representation can be extended to the closed subgroup $\mathbf{B} = \mathbf{MAN}$, the semidirect product of $\mathbf{MA}$ and $\mathbf{N}$, $\mathbf{N}$ being the normal subgroup of $\mathbf{SL}(2,\mathbb{C})$, by the formula

$$\begin{aligned}\pi\begin{pmatrix} \alpha & \gamma \\ 0 & \alpha^{-1} \end{pmatrix} &= \pi_{k,s}(\alpha)\\ &= \pi_k\left(\frac{\alpha}{|\alpha|}\right)|\alpha|^{is}.\end{aligned}$$

Note that

$$\mathbf{MA} = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}_*\right\}$$

is topologically isomorphic with $\mathbb{C}_*$ and $\pi_{k,s}(\alpha)$ are characters of the multiplicative group $\mathbb{C}_*$.

Now, taking the space $\{F : \mathbf{SL}(2,\mathbb{C}) \to \mathbb{C} : F(gman) = \pi_{k,s}(\alpha)^{-1}|\alpha|^{-2}F(g)\}$ with the norm

$$\|F\| = \int\limits_{\mathbf{SU}(2)} |F(k)|^2 dk$$

(well defined by Iwasawa factorization: $\mathbf{SL}(2,\mathbb{C}) = \mathbf{SU}(2)\mathbf{AN}$) and $\mathbf{SL}(2,\mathbb{C})$ acting by

$$\mathcal{R}(\mathbf{B}, k, s, g) F(x) = F(g^{-1}x),$$

the actual Hilbert space and representation is then obtained by completion. This is called the *induced picture.* Its restriction to the subgroup $\mathbf{SU}(2)$ is called the *compact picture.*

Finally, the induced representations in the *noncompact picture* are the irreducible unitary representations of the *principal series*, which will be listed in (29), are obtained by taking the restriction of the induced picture to $\overline{\mathbf{N}} \cong \mathbb{C}$. It follows from the Gauss decomposition (23) given explicitly in (24). For detailed calculations see Example 1 on p. 222 in [22].

5.3 Irreducible representations of $\mathbf{SL}(2,\mathbb{C})$

We list here all irreducible representations of $\mathbf{SL}(2,\mathbb{C})$, referring to [10], [13], or [22] for more details.

The finite-dimensional irreducible representations $\mathcal{P}_g^{m,n}$ of $\mathbf{SL}(2,\mathbb{C})$ associated with the action of $\mathbf{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ are parametrized by pairs of nonnegative integers m and n as follows. For (m,n) let V_{mn} be the vector space of polynomials P in $z_1, z_2, \overline{z}_1, \overline{z}_2$ that are homogeneous of degree m in (z_1,z_2) and homogeneous of degree n in $(\overline{z}_1, \overline{z}_2)$. The action is given by

$$\mathcal{P}_g^{m,n} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

where $g \in \mathbf{SL}(2,\mathbb{C})$. It is the class of smooth irreducible finite representations. The class of holomorphic irreducible finite representations of $\mathbf{SL}(2,\mathbb{C})$ consists of representations $\mathcal{P}_g^m$ (which can be though of as $\mathcal{P}_g^{m,0}$) acting on the vector space V_m of polynomials P in z_1 and z_2 that are homogeneous of degree m in (z_1,z_2).

In turns out that there are no other irreducible finite-dimensional representations of $\mathbf{SL}(2,\mathbb{C})$ up to an equivalence.

For our purpose we shall consider the following realization of V_{mn}. With P in V_{mn} we associate the function $\varphi(\xi)$ by

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1^m \overline{z}_1^n \varphi(z_2/z_1)$$

Then, $\mathcal{P}_g^{m,n}$ has the following realization

$$\mathcal{P}_g^{m,n}\varphi(\xi) = (-b\xi+d)^m \overline{(-b\xi+d)}^n \varphi\left(\frac{a\xi - c}{-b\xi+d}\right). \qquad (25)$$

where according to (5),

$$g^{-1}\cdot\xi = \frac{a\xi - c}{-b\xi + d} \qquad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The infinite-dimensional irreducible representations of $\mathbf{SL}(2,\mathbb{C})$ are classified as follows.

The *nonunitary principal series* of $\mathbf{SL}(2,\mathbb{C})$ is the family of representations $\mathcal{T}_g^{k,w}$ indexed by pairs (k,w) with $k \in \mathbb{Z}$ and $w \in \mathbb{C}$. The representation $\mathcal{T}_g^{k,w}$ is given by

$$\mathcal{T}_g^{k,w}\varphi(\xi) = |-b\xi+d|^{-2+w}\left(\frac{-b\xi+d}{|-b\xi+d|}\right)^k \varphi\left(\frac{a\xi-c}{-b\xi+d}\right) \tag{26}$$

and the Hilbert space is $L^2(\mathbb{C})$ with respect to the measure $\frac{i}{2}(1+|\xi|^2)^{\mathrm{Re}\, w}d\xi d\overline{\xi}$. By comparing (26) with (25) we see that

$$m = \frac{1}{2}(k+w)-1 \tag{27}$$

and

$$n = \frac{1}{2}(-k+w)-1. \tag{28}$$

Moreover, if $k=0$ and w is real, the corresponding representations become unitary for $0<w<2$ with respect to the inner product

$$\langle\varphi,\phi\rangle = \left(\frac{i}{2}\right)^2\int\frac{1}{|\xi-\eta|^{2-w}}\varphi(\xi)\overline{\phi(\eta)}d\xi d\overline{\xi}d\eta d\overline{\eta}$$

and the resulting representations are called *the complementary series.*

The *principal series,* or the *unitary principal series*, of $\mathbf{SL}(2,\mathbb{C})$ is the family of representations (26) which are indexed by (k,is) with $k\in\mathbb{Z}$ and $s\in\mathbb{R}$, that is $\mathcal{T}_g^{k,is}$. In this case

$$\mathcal{T}_g^{k,is}\varphi(\xi) = |-b\xi+d|^{-2+is}\left(\frac{-b\xi+d}{|-b\xi+d|}\right)^k \varphi\left(\frac{a\xi-c}{-b\xi+d}\right). \tag{29}$$

and the representations are unitary with respect to the inner product

$$\langle\varphi,\phi\rangle = \frac{i}{2}\int\varphi(\xi)\overline{\phi(\eta)}d\xi d\overline{\eta}.$$

Moreover, the representations $\mathcal{T}_g^{k,is}$ and $\mathcal{T}_g^{-k,-is}$ are equivalent.

The expressions for the representations $T_g^{k,w}$ in (26) and $T_g^{k,is}$ in (29) are not defined at the points ξ such that $-b\xi + d = 0$. However, the representations are defined on the Hilbert spaces with inner products given by integrals, and in particular, the representations of the principal series $T_g^{k,is}$ are defined on $L^2(\mathbb{C})$, and the formulas hold only up to sets of measure zero.

Up to the equivalence, the trivial representations, the unitary principal series and the complementary series are the only irreducible unitary representations of $\mathbf{SL}(2, \mathbb{C})$, see for example, [13], p. 35. The nonunitary principal series contains all irreducible finite-dimensional representations since $\mathcal{P}^{k,l} \subset \mathcal{T}^{n-m,2+m+n}$. To see this recall that $m = \frac{1}{2}(k+w) - 1$ and $n = \frac{1}{2}(-k+w) - 1$.

6 Harmonic analysis on complex projective line

Fourier analysis on a Lie group $\mathbf{G}$, or on its symmetric space $\mathbf{G}/\mathbf{H}$ where $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$, decomposes the corresponding function spaces into subspaces invariant under irreducible unitary representations induced by the corresponding subgroup of the group $\mathbf{G}$. The classical Fourier analysis on $\mathbb{R}^n$ is a particular case of Euclidean group in which the irreducible unitary representations are generated by its translation subgroup. All representations in this classical case are one-dimensional since the translation subgroup is commutative.

Our projectively adapted Fourier analysis for patterns will be given in terms of the Fourier analysis of the double cover group $\mathbf{SL}(2, \mathbb{C})$ of the projective group $\mathbf{PSL}(2, \mathbb{C})$. The construction of the projectively adapted decomposition of patterns into invariant (under the principal series representations (29)) components will be developed in the next subsections following the general ideas of the harmonic analysis associated with the group $\mathbf{SL}(2, \mathbb{C})$ as originally developed by Gelfand et al [10]. However, the harmonic analysis as formulated here was not presented there.

As it is well known, this $\mathbf{SL}(2, \mathbb{C})$-harmonic analysis involves *only* representations of the principal series, see [10].

6.1 Projective harmonic decomposition

We start by assuming that

$$P = \{f : D \to \mathbb{R}\} \tag{30}$$

is such that D is a compact subset of $\mathbb{C}_*$ where $\mathbb{C}_*$ is the multiplicative group of complex number and associating with the gray-level function f the function h defined as follows. For $\nu \in \Lambda = \{\mu | D \cap D^\mu \neq \emptyset\}$ where $D^\mu = \{\mu\xi | \mu \in \mathbb{C}_*, \xi \in D\}$ we set

$$h\begin{pmatrix} \nu \\ \nu\xi \end{pmatrix} = |\nu|^{-1} f(\nu\xi) \tag{31}$$

Note that the function h has a compact support.

Next, we define the function

$$F(z_1, z_2) = \frac{i}{2}\int h\begin{pmatrix} \mu z_1 \\ \mu z_2 \end{pmatrix} \mu^{-m-1}\overline{\mu}^{\ -n-1} d\mu d\overline{\mu.} \tag{32}$$

and easily verify that

$$F(\lambda z_1, \lambda z_2) = \lambda^m \overline{\lambda}^n F(z_1, z_2).$$

Next, taking $z_1 = \lambda$ and $z_2 = \lambda\xi$ (i.e., the line through $\xi = z_2/z_1$) in the integral (32), we can write

$$F(\lambda, \lambda\xi) = \lambda^m \overline{\lambda}^n \frac{i}{2}\int h\begin{pmatrix} \mu \\ \mu\xi \end{pmatrix} \mu^{-m-1}\overline{\mu}^{\ -n-1} d\mu d\overline{\mu.}$$

It shows that the following integral

$$\Phi(\xi) = \frac{i}{2}\int h\begin{pmatrix} \mu \\ \mu\xi \end{pmatrix} \mu^{-m-1}\overline{\mu}^{\ -n-1} d\mu d\overline{\mu} \tag{33}$$

is constant along the lines $z_2 = \xi z_1$, and therefore, it is a well defined object of the projective space $\widehat{\mathbb{C}}$.

The function $F(z_1, z_2)$, or equivalently, its projective realization $\Phi(\xi)$, is in the space $D_{(m+1,n+1)}$ introduced and discussed in [10], pp. 141-143. These spaces are the natural generalizations of the spaces V_{mn} of homogeneous polynomials on which all finite-dimensional representations of $\mathbf{SL}(2,\mathbb{C})$ are constructed, see Section 5.3. Moreover, the spaces $D_{(m+1,n+1)}$ are fundamental in constructing *all* infinite-dimensional irreducible representations of $\mathbf{SL}(2,\mathbb{C})$, since it can be shown that every such representation, with some natural assumptions, contains (on an everywhere dense subspace with stronger topology) a representation on $D_{(m+1,n+1),}$ see [10], p. 141.

We now prove the following result.

Theorem 12 *Under the transformation*

$$\begin{aligned} T_g h \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= h\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \\ &= h\begin{pmatrix} dz_1 - bz_2 \\ -cz_1 + az_2 \end{pmatrix} \end{aligned}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C}),$$

the function defined by the integral (33) with $m = \frac{1}{2}(k+is) - 1$ *and* $n = \frac{1}{2}(-k+is) - 1$ *and denoted by* $\Phi(\xi; k, s)$; $s \in R$, $k \in Z$, *transforms according to the unitary representation of the principal series* $\mathcal{T}_g^{k,is}$ *of* $\mathbf{SL}(2,\mathbb{C})$ *in (29).*

Proof. *First, we note that*

$$T_g h \begin{pmatrix} \mu \\ \mu\xi \end{pmatrix} = h\begin{pmatrix} \mu(d - b\xi) \\ \mu(-c + a\xi) \end{pmatrix}.$$

Next, by the change of variable $\mu \to \chi = \mu(-b\xi + d)$ *in the integral*

$$\frac{i}{2}\int T_g h \begin{pmatrix} \mu \\ \mu\xi \end{pmatrix} \mu^{-m-1}\overline{\mu}^{\,-n-1} d\mu d\overline{\mu},$$

it can be written, using (33), as follows

$$(-b\xi + d)^m \overline{(-b\xi + d)}^n \Phi\left(\frac{a\xi - c}{-b\xi + d}; k, s\right).$$

Finally, introducing $m = \frac{1}{2}(k+is) - 1$ *and* $n = \frac{1}{2}(-k+is) - 1$ *into the last expression, we obtain*

$$|-b\xi + d|^{-2+is}\left(\frac{-b\xi + d}{|-b\xi + d|}\right)^k \Phi\left(\frac{a\xi - c}{-b\xi + d}; k, s\right)$$

which is the unitary principal series $\mathcal{T}_g^{k,is}$ *of* $\mathbf{SL}(2,\mathbb{C})$ *acting on* $\Phi(\xi;k,s)$*. In fact, using the norm* $||f||_2^2 = \frac{i}{2}\int|f(z)|^2dzd\overline{z}$ *we have*

$$\left\|\mathcal{T}_g^{k,is}\Phi(\xi;k,s)\right\|_2 = \left\||d-b\xi|^{-2}\Phi\left(g^{-1}\cdot\xi;k,s\right)\right\|_2.$$

Now, since $d\xi'd\overline{\xi}' = |d-b\xi|^{-4}d\xi d\overline{\xi}$ *where* $\xi' = g^{-1}\cdot\xi = (-c+a\xi)/(d-b\xi)$*, we easily obtain that* $\left\|\mathcal{T}_g^{k,is}\Phi\right\|_2 = \|\Phi\|_2$. ■

Note that the unitary representation $\mathcal{T}_g^{k,is}$, defined in Theorem 12 on the function space $D_{(m+1,n+1)}$ where $m = \frac{1}{2}(k+is)-1$ and $n = \frac{1}{2}(-k+is)-1$, which is isometry on $D_{(m+1,n+1)}$ by this theorem, can be extended uniquely to unitary operators on the Hilbert space obtained by the completion of $D_{(m+1,n+1)}$ with respect to the norm $||f||_2$ on $L^2(\mathbb{C})$. This leads to the definition of the principal series of $\mathbf{SL}(2,\mathbb{C})$ acting on $L^2(\mathbb{C})$, which was given in (29).

Theorem 13 *The function* f *is given in terms of* $\Phi(\xi;k,s)$ *by*

$$f(\xi) = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi;k,s)ds \tag{34}$$

Proof. *This theorem can be proven as follows. First, by introducing* $\mu = e^{u+i\theta}$ *into (33) with* f *given in (31), we can express it as the standard Fourier integral*

$$\Phi(\xi;k,s) = \int_{-\infty}^{\infty}\int_{0}^{2\pi}e^u f(e^{u+i\theta}\xi)e^{-i(us+\theta k)}d\theta du. \tag{35}$$

Using that f *has a compact support and* 0 *is not in its domain* D*, the inverse of (35) is given by*

$$e^u f(e^{u+i\theta}\xi) = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi;k,s)e^{i(us+\theta k)}ds. \tag{36}$$

Finally, by taking $u=\theta=0$ *in (36) we obtain (34).* ■

Thus, Theorem 12 and Theorem 13 give $\mathbf{SL}(2,\mathbb{C})$-*analogue of the Fourier decomposition.* We elaborate further on this point by identifying, what we call, the projective Fourier transform and Plancherel's theorem.

6.2 Projective Fourier transform of a pattern

For a given pattern (30) the integral 33, which can be also written as

$$\Phi(\xi;k,s)=\frac{i}{2}\int|\mu|^{-1}f(\mu\xi)\left(\frac{\mu}{|\mu|}\right)^{-k}|\mu|^{-is}d\mu d\overline{\mu} \tag{37}$$

where $\mu\neq 0$, transforms under the principal series $\mathcal{T}_g^{k,is}$ of $\mathbf{SL}(2,\mathbb{C})$ given in (29) and the function f has the decomposition (Theorem 13)

$$f(\xi)=(2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi;k,s)ds.$$

Introducing $\zeta=\mu\xi$ into the integral in (37), we obtain after simple calculation

$$\Phi(\xi;k,s)=|\xi|^{is-1}\left(\frac{\xi}{|\xi|}\right)^{k}\mathfrak{F}(k,s) \tag{38}$$

and hence,

$$f(\xi)=(2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|\xi|^{is-1}\left(\frac{\xi}{|\xi|}\right)^{k}\mathfrak{F}(k,s)ds \tag{39}$$

and

$$\mathfrak{F}(k,s)=\frac{i}{2}\int f(\zeta)\left(\frac{\zeta}{|\zeta|}\right)^{-k}|\zeta|^{-is-1}d\zeta d\overline{\zeta} \tag{40}$$

is the *projective analogue* of the classical Fourier transform pair,

$$f(x)=(2\pi)^{-2}\int F(k)e^{i<x,k>}dk$$

and

$$F(k)=\int f(x)e^{-i<x,k>}dx$$

where $<\cdot,\cdot>$ is the scalar product in $\mathbb{R}^2$.

Note that

$$\pi_{k,s}(\zeta) = \left(\frac{\zeta}{|\zeta|}\right)^k |\zeta|^{is} \tag{41}$$

are the *characters* of the Borel subgroup of $\mathbf{SL}(2,\mathbb{C})$, see [12], the projective analogue of the characters $e^{i<x,k>}$ of the translation subgroup of Euclidean group. Because of this analogy we give the following definition.

Definition 14 *$\mathfrak{F}(k,s)$ in (40) is called the projective Fourier transform of the pattern $P = \{f : D \to \mathbb{R}\}$.*

We end here by showing in Figure 1 some graphs of the real and imaginary parts of the characters $\pi_{k,s}(\zeta)$.

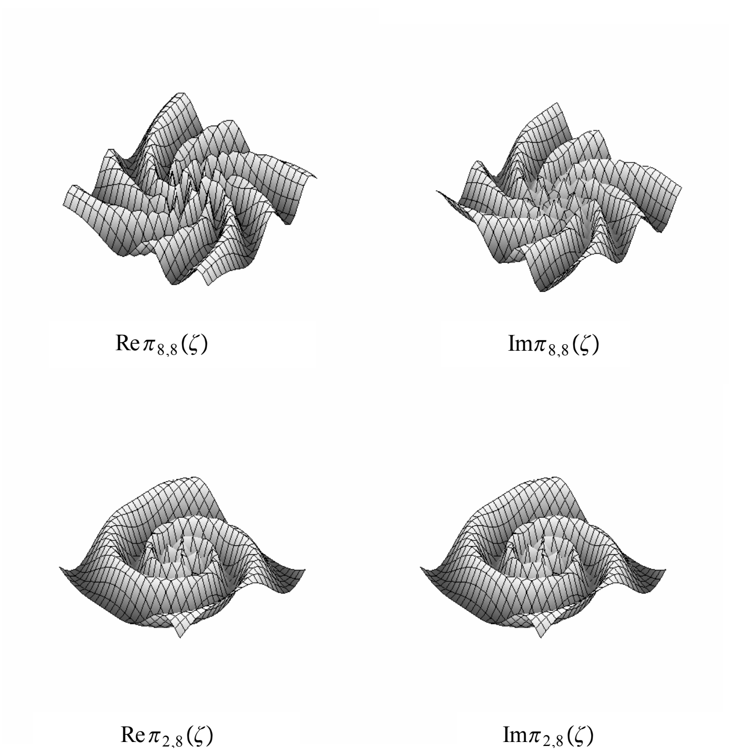

Figure 1. Graphs of the real and imaginary parts of the characters $\pi_{k,s}(\zeta)$.

6.3 Plancherel's theorem

Plancherel's theorem also holds in this projective analogue. To this end, we define the measure ω by

$$\int \mathfrak{F}(k,s)d\omega = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathfrak{F}(k,s)ds$$

and the norm $||\mathfrak{F}(k,s)||$ as follows

$$||\mathfrak{F}(k,s)||^2 = \int |\mathfrak{F}(k,s)|^2 d\omega.$$

Finally, Plancherel's theorem can be formulated as follows.

Theorem 15 *Let f be in $L^2(\mathbb{C})$. Then,*

$$||f||_2 = ||\mathfrak{F}(k,s)|| \tag{42}$$

Proof. *We start by assuming that the function $f \in L^2(\mathbb{C})$ vanishes at the origin. In this case, using Plancherel's theorem for the (standard) Fourier transform in (35), we obtain*

$$\int_{-\infty}^{\infty}\int_{0}^{2\pi} e^{2u}|f(e^{u+i\theta}\xi)|^2 d\theta du = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|\xi|^{-2}|\mathfrak{F}(k,s)|^2 ds. \tag{43}$$

Writing $e^{u+i\theta}\xi = z$, we have $\frac{i}{2}dzd\overline{z} = e^{2u}|\xi|^2 d\theta du$. Using it in (43), we arrive at

$$\frac{i}{2}\int |f(z)|^2\,|\xi|^{-2}dzd\overline{z} = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathfrak{F}(k,s)|^2|\xi|^{-2}ds.$$

It shows that (42) holds for f with bounded support and which vanishes at the origin. However, these functions form an everywhere dense set in $L^2(\mathbb{C}_*)$ which shows that (42) holds also for $f \in L^2(\mathbb{C})$. (We have used the fact that f can be modified on any set of measure zero.) ■

Thus, the last theorem says that the mapping $f(\xi) \to \mathfrak{F}(k,s)$ is *isometric* with respect to the norms $||f||_2$ and $||\mathfrak{F}||$.

In our final remark of this section we comment on a similar exposition of the corresponding inversion and Plancherel formulas. In an excellent monograph [20], the author decomposes the (right) regular representations of $\mathbf{SL}(2,\mathbb{C})$ (cf, Example 11 given in Section 5.1) via the spectral decomposition of the complex dilations (that are commuting with the representations). However, it seems from the brief discussion given in [20] on pp. 220-221 that the restriction of the representations taken in the noncompact picture, and given in (2.95) in [20], are not the principal series representations. In fact, they are not unitary representations with the exponents λ and μ given in (2.94) which, in the notation used here, read $m = \frac{1}{2}(k + is - 1)$ and $n = \frac{1}{2}(-k + is - 1)$, respectively. They should be compared with (27) and (28) where $w = is$. On the other hand, if one takes the principal series representation then, it seems that the Plancherel's formula in [20] will not hold.

7 The conclusion: Projectively adapted pattern representation

There are two distinct ways of representing and rendering pictures in machine vision. The geometric way begins with the representation of shapes of objects in scenes in terms of geometrical primitives (mathematical abstractions), such as lines, polygons, splines, cylinders and spheres. Here theorems from analytical geometry (both Euclidean and projective) are of paramount importance. The imaging way, on the other hand, deals with arrays of numbers - discrete samples of pixels - that are coming mostly from non-geometrical sources such as digitized satellite photographs or X-radiographs, for example. Here the Fourier and wavelet theories, with the Sampling Theorem, are of paramount importance.

Of course, both approaches are interrelated. For example, one can extract geometric data from sampled data. However, it frequently introduces thresholding artifacts and in many situations, such as medical diagnostic imaging, is highly undesirable.

The cameras used in computer vision research that have been mentioned in Section 4 belong to the geometric approach. They are used to extract geometric information (for example, projective invariants) from scenes, and in this sense, they play an active role, see, [2], [16], [17] and [18] for some recent papers. On the other hand, the $\mathbf{PSL}(2, \mathbb{C})$-camera model justifies the word "projective", but otherwise plays a passive role. The projective analogue of Fourier decomposition for patterns based on the $\mathbf{PSL}(2, \mathbb{C})$-camera model, as it has been constructed in this work, belongs to the imaging approach.

7.1 Projectively adapted properties of the pattern decompositions

We shall discuss a closed-form relation under projective distortions of the representation of patterns given in terms of the projective Fourier transforms.

First, we recall that for a given pattern $P = \{f : D \to \mathbb{R}\}$, where $D \subset \mathbb{C}_*$ is compact, we have constructed its representation (39) in terms of the characters (41) of the Borel subgroup $\mathbf{B} = \mathbf{MAN}$ of the group $\mathbf{SL}(2, \mathbb{C})$ with the coefficients of the decomposition (40) given in terms of the projective Fourier transform $\mathfrak{F}(k, s)$ of f. Moreover, this $\mathfrak{F}(k, s)$ can be written in the form of the standard Fourier transform,

$$\mathfrak{F}(k, s) = \int_{-\infty}^{\infty} \int_{0}^{2\pi} e^{u} f(e^{u+i\theta}) e^{-i(us+\theta k)} d\theta du \tag{44}$$

which is obtained from (35) by taking $\xi = 1$, which should by important for developing the fast algorithms for the projective Fourier transform.

Next, recalling the $\mathbf{PSL}(2, \mathbb{C})$-camera model, the transformations of a pattern consist of translations in the camera plane ($\overline{\mathbf{N}}$), "virtual" perpendicular translations out of the camera plane ($\mathbf{A}$) and rotations ($\mathbf{K} = \mathbf{SU}(2)$, including also rotations in the camera plane given by $\mathbf{M} \subset \mathbf{K}$). Iterations of these transformations, included in the camera model, can be produced by a camera of a plane flying over a terrain and taking pictures of the same scene.

Under the action of

$$g \in \mathbf{KA\overline{N}}$$

the pattern's intensity function $f(z)$ transforms as $f(g^{-1} \cdot z)$, and consequently, we obtain from (39) the projectively transformed Fourier reconstruction

$$f(g^{-1}\cdot z) = (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|g^{-1}\cdot z|^{is-1}\left(\frac{g^{-1}\cdot z}{|g^{-1}\cdot z|}\right)^{k}\mathfrak{F}(k,s)ds \tag{45}$$

where $\mathfrak{F}(k,s)$ is the projective Fourier transform of the original (undistorted) pattern, giving a closed-form relation.

The analogy with the classical Fourier transform is obtained by taking

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathbf{MA}. \tag{46}$$

For this subgroup $\mathbf{MA}$, the subgroup of $\mathbf{SL}(2,\mathbb{C})$ with unitary representations given by the characters (41), we have,

$$\begin{aligned}
f(g^{-1}\cdot z) &= f(a^2 z) \\
&= (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|a^2 z|^{is-1}\left(\frac{a^2 z}{|a^2 z|}\right)^{k}\mathfrak{F}(k,s)ds \\
&= (2\pi)^{-2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|z|^{is-1}\left(\frac{z}{|z|}\right)^{k}|a^2|^{is-1}\left(\frac{a^2}{|a^2|}\right)^{k}\mathfrak{F}(k,s)ds.
\end{aligned}$$

Thus, we conclude that under the transformation

$$f(z) \longmapsto f(g^{-1}\cdot z)$$

where g is given in (46), the projective Fourier transform $\mathfrak{F}(k,s)$ transforms as follows

$$\mathfrak{F}(k,s) \longmapsto |a^2|^{is-1}\left(\frac{a^2}{|a^2|}\right)^{k}\mathfrak{F}(k,s).$$

Also, our projectively adapted representation is invertible, that is, it preserves all information of the pixel gray-level function of the original pattern (Theorem 13 and Theorem 15). Usually, images from satellite and medical images are stored by techniques which preserve information and some countries are already investigating legal requirements for the kind of techniques can be used for archiving medical images, see [14].

7.2 Computer simulations

We start with the first test in which we reconstruct a polar chessboard pattern shown in Figure 2.

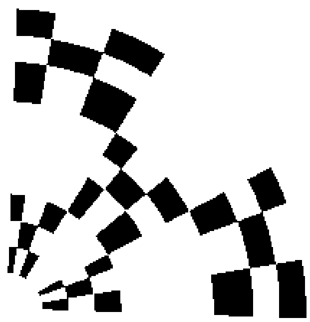

Figure 2. Two-scale chessboard pattern.

The reconstruction of this pattern is obtained using the inverse projective Fourier transform (39) with the projective Fourier transform given in (40). The approximations we have used in order to perform the computer simulations consisted of the summation from $-N$ to N and the integration from $-S$ to S taken in (39). In the reconstruction shown in Figure 3, we have taken $N = 64$ and $S = 220$ and used the densityplot, a Maple procedure, running the Maple software on a 200 Mhz Pentium Pro® PC.

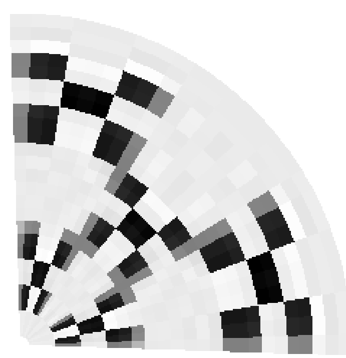

Figure 3. Computer simulation - Maple procedure - N=64 and S=220.

The second test demonstrates the projectively adapted properties of the projective Fourier transform. Here, the pattern consisting of a circular ring is reconstructed in Figure 4 by the same technique used in the first test.

Fig4

The pattern is next projectively distorted by applying

$$g = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

which represents rotation (out of the image plane $x_2 = 1$) by the angle $\phi/2$ about x_3-axis, of the ring pattern and projecting it back on the image plane. Computer simulations of (45) with

$$g^{-1} \cdot z = \frac{z\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}}{-iz\sin\frac{\phi}{2} + \cos\frac{\phi}{2}}, \tag{47}$$

using the same method as before, are shown in the Figure 5 for $\phi = \frac{\pi}{12}$, in Figure 6 for $\phi = 0.378$, in Figure 7 for $\phi = \frac{\pi}{6}$ and finally in Figure 8 for $\phi = 0.55$. All distorted images are shown in the same scale as the original pattern in Figure 4.

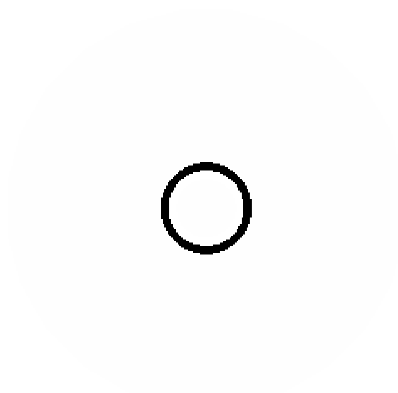

Figure 4. Circular ring pattern.

Figure 5. Computer simulation - $\phi = \frac{\pi}{12}$

Figure 6. Computer simulation - for $\phi = 0.378$.

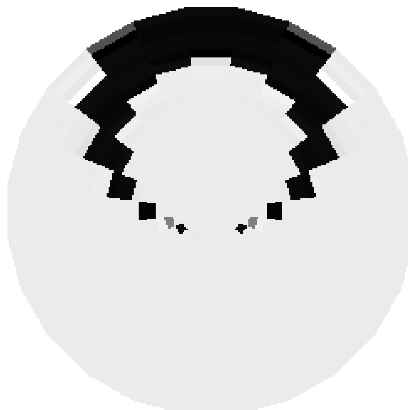

Figure 7. Computer simulation - $\phi = \frac{\pi}{6}$.

Figure 8. Computer simulation - $\phi = 0.55$.

Thus, we have produced the pattern's projective distortions from the only one projective Fourier transform of the original pattern.

Note that the projective Fourier transform of the ring pattern is given by a single integral, as the pattern is symmetric with respect to the angular variable, and therefore its reconstruction is easily obtained running the densityplot procedure. On the other hand, its projective distortions produced by (47), when expressed in terms of their projective Fourier transforms, are given by double integrals. We were unable to simulate any pattern reconstruction involving double integrals that cannot be simplified at least to a product of single integrals. For example, the projective Fourier transform of the polar chessboard pattern involved the product of two single integrals.

The projectively adapted characteristics of the projective Fourier transform allow us to obtain *any* projective distortion of a pattern, as produced when the pictures of the same pattern are taken by a camera from different vantage points, using *only* the projective Fourier transforms of the original pattern.

Finally, we remark that the projective Fourier transform has an expression in terms of the standard Fourier transform (35). It should be of importance in practical implementations of this projective harmonic analysis in image processing. In particular, one can try to adapt the fast Fourier transform algorithms to the projective analogue. We have already done some work in this direction.

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