Exponential and Logarithmic Functions

Exponential Functions

Example

Suppose you are a salaried employee, that is, you are paid a fixed sum each pay period no matter how many hours you work. Moreover, suppose your union contract guarantees you a 5% cost-of-living raise each year. Then your annual salary is an increasing function of the number of years you have been employed, because your annual salary will increase by some amount each year. However, the amount of the increase is different from year to year, because as your salary increases, the amount of your 5% raise increases too. This phenomenon is known as *compounding*.

Example

Assume your starting salary is $28,000 per year. Let $S(t)$ be your annual salary after full years of employment. Therefore, $S(0)$ is interpreted to mean your initial salary of $28,000. How can we evaluate $S(1)$, your salary after 1 year of employment? Since your salary is increasing by 5% each year, this means $S(1)$ is 5% more than $S(0)$. In other words, $S(1)$ is 105% of $S(0)$. Thus, we can evaluate $S(1)$ as shown here, by changing the percentage 105% to a decimal number:

$S(1) = 105\%$ of $S(0) = 1.05 \times S(0) = 1.05 \times 28000$

$S(2) = 105\%$ of $S(1) = 1.05 \times S(1) = 1.05^2 \times 28000$

$S(3) = 105\%$ of $S(2) = 1.05 \times S(2) = 1.05^3 \times 28000$

$S(4) = 105\%$ of $S(3) = 1.05 \times S(3) = 1.05^4 \times 28000$

$S(5) = 105\%$ of $S(4) = 1.05 \times S(4) = 1.05^5 \times 28000$

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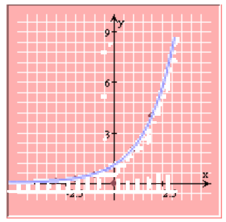
$S(t) = 1.05^t \times 28000$

Graph of Exponential Functions

$f(x) = 2^x$

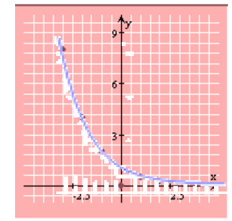
x	2^x
–3	1/8
–2	1/4
–1	1/2
0	1
1	2
2	4
3	8



Graph of Exponential Functions

$f(x) = \left(\frac{1}{2}\right)^x$

x	2^x
–3	8
–2	4
–1	2
0	1
1	1/2
2	1/4
3	1/8



Exponential Functions

Exponential functions have symbol rules of the form

$$f(x) = c \cdot b^x$$

b: base or growth factor -- must be positive real number but cannot be 1, i.e. $b > 0$ and $b \neq 1$

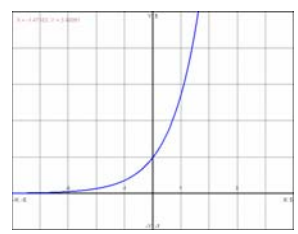
c: coefficient greater than 0

the domain of *f* is $(-\infty, \infty)$

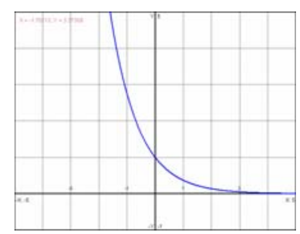
the range of *f* is $(0, \infty)$

Natural Exponential Function

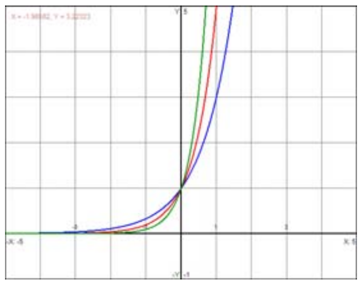
$f(x) = e^x$



$f(x) = e^{-x}$



Example

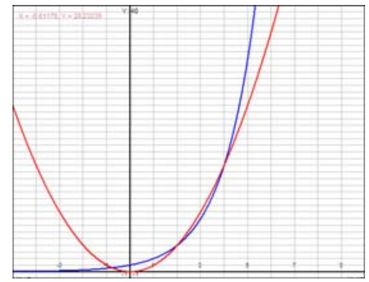


$f(x) = 3^x$

$f(x) = 5^x$

$f(x) = 10^x$

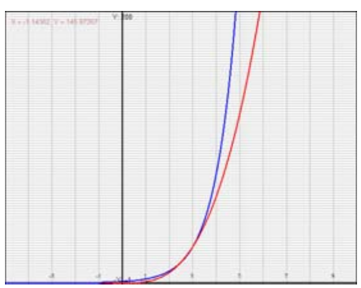
Example



$f(x) = 2^x$

$f(x) = x^2$

Example

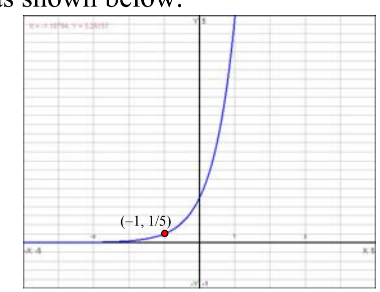


$f(x) = 3^x$

$f(x) = x^3$

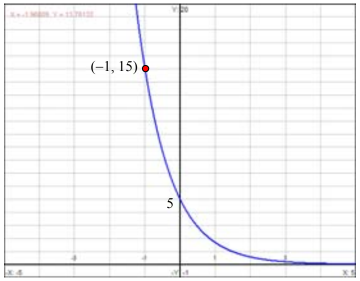
Problem 10 on page 344

Find the exponential function $f(x) = a^x$ whose graph as shown below.



Problem 34 on page 344

Find the exponential function $f(x) = ca^x$ whose graph as shown below.



Practice Problems on page 356

3,5,7,8,11,13-22,23-30,35-38,39-44

Logarithmic Functions

Logarithmic Functions

Consider the exponential function f shown here with base $b = 2$ and initial value $c = 1$.

$$f(x)= 2^x$$

Suppose we want to find the input number for that matches the output values 8 and 15, in other words, we want to solve the equation

$$8 = 2^x \text{ and } 15 = 2^x$$

Logarithmic Functions

Let's introduce a new function designed to help us express solutions to equations like the two shown here, which are solved by finding particular input numbers for the exponential function *f*. We give this new function a special label:

$$log_2$$

Logarithmic Functions

log_2 helps us express inputs for the function *f*. Thus, for example, we evaluate $log_2 8 = 3$, because $f(3) = 2^3 = 8$. Likewise, we evaluate

$log_2 4 = 2$, because $f(2) = 2^2 = 4$

$log_2 32 = 5$, because $f(5) = 2^5 = 32$

$log_2 1 = 0$, because $f(0) = 2^0 = 1$

$Log_2 1/2 = -1$, because $f(-1) = 2^{-1} = ½$

In general,

$log_2 y = x$, because $f(x) = 2^x = y$

That is exponential function and logarithmic function are inverse of each other.

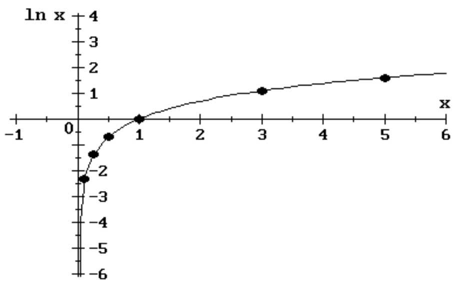
Common and Natural Logarithms

- A *common logarithm* is a logarithm with base 10, log_{10}.
- A *natural logarithm* is a logarithm with base *e*, *ln*.

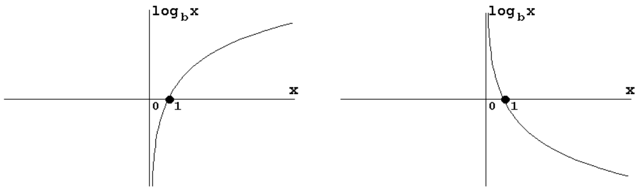
Properties of Logarithms

1. $log_a 1 = 0$
2. $log_a a = 1$
3. $log_a a^x = x$
4. $a^{\log_a x} = x$

Graphs of Logarithmic Functions

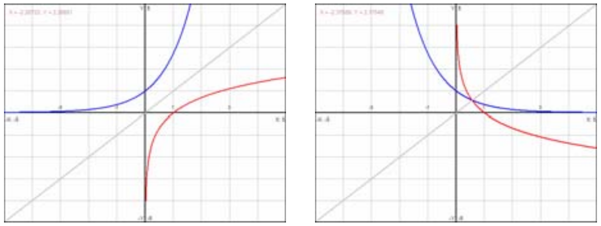


Graphs of Logarithmic Functions



Graphs of Logarithmic Functions

$f(x) = e^x$ $f(x) = \ln x$ $f(x) = e^{-x}$ $f(x) = -\ln x$



Practice Problems on page 356

2,3,7,8,9,11,12,13-18,19,20,21,22,23,33,72

Laws of Logarithms

Change of Base

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Problem 50 page 363

Evaluate $\log_2 5$

Practice Problems on page 363

49,51,53

Exponential and Logarithmic Equations

Exponential Equations

Problems on page 372

2. $10^{-x} = 2$
6. $3^{2x-1} = 5$
20. $10^{1-x} = 6^x$
32. $e^{2x} - e^x - 6 = 0$

Logarithmic Equations

Problems on page 373

38. $\log(x-4) = 3$
40. $\log_3(2-x) = 3$
42. $\log_2(x^2 - x - 2) = 2$

Compound Interest

If P is a principal of an investment with an interest r for a period of t years, then the amount A of the investment is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Problem 56 on page 373

A man invests \$4000 in saving certificates that bear an interest rate of 9.75% peer year, compounded semiannually. How long a time period should she choose in order to save an amount of \$5000?

$A = 5000$

$P = 4000$

$r = .0975$

$n = 2$

$$5000 = 4000\left(1+\frac{.0975}{2}\right)^{2t}$$

$$5000 = 4000(1.04875)^{2t}$$

$$(1.04875)^{2t} = 1.25$$

$$2t = \frac{\ln 1.25}{\ln 1.04875} = \frac{0.2231}{0.0476} = 4.6870$$

$$t = 2.3435 \Rightarrow 2 \text{ years and } 4 \text{ months}$$

Problem 59 on page 373

How long will it take for an investment of \$1000 to double in value if the interest rate is 8.5% per year, compounded quarterly.

$A = 2000$

$P = 1000$

$r = .085$

$n = 4$

$$2000 = 1000\left(1+\frac{.085}{4}\right)^{4t}$$

$$2000 = 1000(1.02125)^{4t}$$

$$(1.02125)^{4t} = 2$$

$$4t = \frac{\ln 2}{\ln 1.02125} = \frac{0.6931}{0.0210} \approx 33$$

$$t = 8.25 \Rightarrow 8 \text{ years and } 3 \text{ months}$$

Practice Problems on page 372

1,3,5,9,15,19,27,31,35,39,41,51,53,55,57

Modeling with Exponential and Logarithmic Functions

Exponential Growth Model

A population that experiences *exponential growth* increases according to the model

$$n(t) = n_0 e^{rt}$$

where

$n(t)$ = population at time t

n_0 = initial size of population

r = relative rate of growth

t = time

Problem 2 page 386

The number of a certain species of fish is modeled by the function

$$n(t) = 12e^{0.012t}$$

where t is measured in years and $n(t)$ is measured in millions.

a) What is the relative rate of growth of the fish population? Express your answer in percentage.
b) What will the fish population be after 5 years?
c) After how many years will the number fish reach 30 million?

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the remaining mass of radioactive at time t is modeled by

$$m(t) = m_0 e^{-rt}$$

where

$m(t)$ = remaining mass of radioactive at time t

$r = \frac{\ln 2}{h}$

t = time

Problem 14 page 387

The half-life of radium-226 is 1600 years. Suppose we have a 22-mg sample.

a) Find a function that models the mass remaining after t years.
b) How much of the sample will remain after 4000 years?
c) After how long will only 18 mg of the sample remain?

Practice Problems on page 386

1,3,5,15,17,19