Sequences and Series

Sequences and Summation Notation

Sequences

A *Sequence* is a set of numbers written a specific order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

a_1 is called the *first term*, a_2 is the *second term* and a_n is the *general term*.

Examples:

1,3,5,7,9,…

1,4,9,16,25,36,…

Definition of a Sequence

- A *sequence* is a function f whose domain is the set of natural numbers. The values $f(1), f(2), f(3), \ldots$ are the *terms* of the sequence.
- Example:

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	…	$f(n) = n^2$	…
1	4	9	16	25	…		…
a_1	a_2	a_3	a_4	a_5	…	a_n	…

Problems on page 830

Find the first four terms and the 100th term of the sequence

2. $a_n = 2n + 3$

6. $a_n = \dfrac{1}{n^2}$

8. $a_n = (-1)^{n+1}\dfrac{n}{n+1}$

Problems on page 831

Find the nth term of a sequence whose several terms are given.

24. $-\dfrac{1}{3}, \dfrac{1}{9}, -\dfrac{1}{27}, \dfrac{1}{81}, \ldots$

26. $5, -25, 125, -625, \ldots$

28. $\dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \dfrac{6}{7}, \ldots$

Recursively Defined Sequences

Problems on p. 830

Find the first five terms of the given recursively defined sequence.

12. $a_n = \dfrac{a_{n-1}}{2}$ and $a_1 = -8$

16. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_1 = a_2 = a_3 = 1$

The Partial Sums of a Sequence

For the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The partial sums are

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Examples

Problems on p. 831

Find the first six partial sums S_1, S_2, S_3, S_4, S_5, S_6 of the sequence.

32. $1^2, 2^2, 3^2, 4^2, \ldots$

34. $-1, 1, -1, 1, \ldots$

Find the first four partial sums and the nth of the sequence a_n.

36. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

Sigma Notation

Given the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The sum of the first n terms can be written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Examples

Problems on p. 831

Write the sum without using sigma notation.

54. $\sum_{i=0}^{4} \frac{2i-1}{2i+1}$

56. $\sum_{k=6}^{9} k(k+3)$

Write the sum using sigma notation.

60. $2 + 4 + 6 + \cdots + 20$

64. $\frac{\sqrt{1}}{1^2} + \frac{\sqrt{2}}{2^2} + \frac{\sqrt{3}}{3^2} + \cdots + \frac{\sqrt{n}}{n^2} +$

Properties of Sums

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every positive integer n and any real number c, the following properties hold

1. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof:

Practice Problems on Page 830

1,3,5,7,11,13,15,23,25,27,29,31,33,35,39,41,45,53,55,59,65,69

Arithmetic Sequences

Definition of an Arithmetic Sequence

An *Arithmetic Sequence* is a sequence of the form:

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \ldots$$

a is the *first term* and d is the *common difference* of the sequence.

The *n*th term of an arithmetic sequence is given by

$$a_n = a + (n - 1)d$$

Examples

Problems on page 837

Determine whether the sequence is arithmetic . If it is arithmetic, find the common difference.

10. $3, 6, 9, 13, \ldots$
12. $2, 4, 6, 8, \ldots$
14. $\ln 2, \ln 4, \ln 8, \ln 16, \ldots$
16. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Examples

Problems on page 837

A sequence is given.

a) Find the first five terms of the sequence.
b) What is the common difference d?
c) Graph the terms found in part a).

2. $a_n = 3 - 4(n - 1)$

4. $a_n = \frac{1}{2}(n - 1)$

Examples

Problems on page 837

Find the nth term of the arithmetic sequence with given first term a and the common difference d. What is the 10^{th} term?

6. $a = -6,\ d = 3$

8. $a = \sqrt{3}\ ,\ d = \sqrt{3}$

Finding Terms of an Arithmetic Sequence

Problems on page 837

34. The 12^{th} term of an arithmetic sequence is 32, and the fifth term is 18. Find the 20^{th} term.
36. The 100^{th} term an arithmetic sequence is 98, and the common difference is 2. Find the first three terms.
38. The first term of an arithmetic sequence is 1, and the common difference is 4. Is 11,937 a term of this sequence? If so, which term is it?

Partial Sums of an Arithmetic Sequence

Suppose we to find the sum of the numbers

$$1, 2, 3, 4, \ldots, 100$$

That is $\sum_{k=1}^{100} k$

$$S = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$

$$\underline{S = 100 + 99 + 98 + \cdots + 3 + 2 + 1}$$

$$2S = 101 + 101 + 101 + \cdots + 101 + 101 + 101$$

$$S = 101 \times 100 / 2 = 10100 / 2 = 5050$$

Partial Sums of an Arithmetic Sequence

Suppose we to find the sum of the arithmetic sequence

$a, a+d, a+2d, a+3d, \ldots, a+(n-2)d, a+(n-1)d$

That is to find $S_n = \sum_{k=1}^{n}[a+(k-1)d]$

$$\begin{array}{lllll} S_n = & a & + \; a+d & + \cdots & + \; a+(n-2)d + a+(n-1)d \\ S_n = & a+(n-1)d + a+(n-2)d & & + \cdots & + \; a+d \; + \; a \\ \hline 2S_n = & 2a+(n-1)d + a+(n-1)d & & + \cdots & + \; a+(n-1)d + a+(n-1)d \end{array}$$

$2S_n = n\,[2a+(n-1)d]$

$S_n = \frac{n}{2}[a+(k-1)d]$

Partial Sums of an Arithmetic Sequence

For the arithmetic sequence $a_n = a + (n-1)d$, the nth partial sum

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + [a+(n-1)d]$$

is given by either of the following formulas.

1. $S_n = \frac{n}{2}[2a+(n-1)d]$

2. $S_n = n\left(\frac{a+a_n}{2}\right)$

Finding Partial Sums of an Arithmetic Sequence

Problems on page 837

Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions.

40. $a = 3$, $d = 2$, $n = 12$

42. $a = 100$, $d = -5$, $n = 8$

Problems on page 837

Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions

46. $-3 + (-1/2) + 0 + 3/2 + 3 + \cdots + 30$

48. $-10 - 9.9 - 9.8 - \cdots - 0.1$

50. $\sum_{n=0}^{20}(1-2n)$

Problems on page 837

56. An arithmetic sequence has the first term $a_1 = 1$ and the forth term $a_4 = 16$. How many terms of this sequence must be added to get 2700?

60. Drive-in Theater A drive-in theater has spaces for 20 cars in the first parking row, 22 in the second, 24 in the third, and so on. If there are 21 rows in the theater, find the number of cars that can be parked.

Practice Problems on Page 837

1,3,5,7,11,13,17,19,21,23,25,27,33,35,37,39,41,43, 45,47,49,55,59

Geometric Sequences

Definition of an Geometric Sequence

A *Geometric Sequence* is a sequence of the form:

$$a,\ a\,r,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

a is the *first term* and r is the *common ratio* of the sequence.

The nth term of an arithmetic sequence is given by

$$a_n = ar^{n-1}$$

Examples

Problems on page 844

Determine whether the sequence is geometric . If it is geometric, find the common ratio.

10. $2, 6, 18, 36, \ldots$
12. $27, -9, 3, -1, \ldots$
14. $e^2, e^4, e^6, e^8, \ldots$
16. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

Examples

Problems on page 844

A sequence is given.

a) Find the first five terms of the sequence.
b) What is the common ratio r?
c) Graph the terms found in part a).

2. $a_n = 3(-4)^{n-1}$

4. $a_n = 3^{n-1}$

Examples

Problems on page 844

Find the nth term of the arithmetic sequence with given first term a and the common difference d. What is the 10th term?

6. $a = -6,\ r = 3$

8. $a = \sqrt{3}\ ,\ r = \sqrt{3}$

Finding Terms of a Geometric Sequence

Problems on page 844

34. The first term of a geometric sequence is 3, and the third term is 4/3. Find the fifth term.
36. The common ratio in a geometric sequence is 3/2, and the fifth term is 1. Find the first three terms.
38. The second and the fifth terms of a geometric sequence are 10, and 11250, respectively. Is 31,250 a term of this sequence? If so, which term is it?

Partial Sums of a Geometric Sequence

Suppose we to find the sum of the numbers

1, 2, 4, 8, 16, 32, 64, 128, 256, 512

That is $\sum_{k=1}^{10} 2^{k-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9$

$$\begin{aligned}
S &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 \\
2S &= \phantom{2^0 +{}} 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} \\
\hline
S - 2S &= 2^0 - 2^{10} \\
(1-2)S &= 2^0 - 2^{10} \\
S &= (2^0 - 2^{10}) / (1-2)
\end{aligned}$$

Partial Sums of a Geometric Sequence

Suppose we to find the partial sum of the geometric sequence $a_n = ar^{n-1}$

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

That is to find $S_n = \sum_{k=1}^{n} ar^{k-1} \quad r \neq 1$

$$\begin{aligned}
S_n &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1} \\
rS_n &= \phantom{a + {}} ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1} + ar^n \\
\hline
S_n - rS_n &= a \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - ar^n \\
(1 - r)S_n &= a - ar^n \\
S_n &= (a - ar^n) / (1 - r) \\
S_n &= a\frac{1-r^n}{1-r}
\end{aligned}$$

Partial Sums of an Arithmetic Sequence

For the arithmetic sequence $a_n = ar^{n-1}$, the nth partial sum

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1} \quad (r \neq 1)$$

is given by either of the following formulas.

$$S_n = a\frac{1-r^n}{1-r}$$

Finding Partial Sums of a Geometric Sequence

Problems on page 844

Find the partial sum S_n of the Geometric sequence that satisfies the given conditions.

40. $a = 2/3$, $r = 1/3$, $n = 4$

42. $a = 0.12$, $d = 0.00096$, $n = 4$

Problems on page 844

Find the sum.

44. $1 - 1/2 + 1/4 - 1/8 + \cdots - 1/512$

46. $\sum_{j=0}^{5} 7\left(\frac{3}{2}\right)^j$

Infinite Geometric Series

An *infinite geometric series* is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1} + \cdots$$

The sum is $S_n = a\frac{1-r^n}{1-r} \quad (r \neq 1)$

If $|r| < 1$, then as $n \to \infty$, $r^n \to 0$.

Hence

$$S_n = \frac{a}{1-r} \quad \text{as } n \to \infty$$

Infinite Geometric Series

If $|r| < 1$, then the *infinite geometric series*

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1} + \cdots$$

has the sum

$$S = \frac{a}{1-r}$$

Finding the Sum of an Infinite Geometric Series

Problems on page 844

Find the sum of the infinite geometric series.

48. $1 - 1/2 + 1/4 - 1/8 + \cdots$

50. $2/5 + 4/25 + 8/125 + \cdots$

Problems on page 845

Express the repeating decimal as fraction.

56. $0.2\overline{53}$

60. 0.123123123…

Problems on page 845

70. Geometric Savings Plan A very patient woman wishes to become a billionaire. She decides to follow a simple scheme: She puts aside 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the number of cents each day. How much money will she have at the end of 30 days? How many days will it take this woman to realize her wish?

Practice Problems on Page 844

1,3,5,7,11,13,17,19,21,23,27,33,35,37,39,41,43, 45,47,49,51,55,57,59,67,69

Mathematics of Finance

What is an Annuity?

- An *Annuity* is a sum of money that is paid in regular equal payments.
- The *amount of an Annuity* is a sum of all the individual payments from the time of the first payment until the last payment is made together with all the interest.

Example 1 page 848

An investor deposits $400 every December 15 and June 15 for 5 years in an account that earns interest at the rate of 8% per year, compounded semiannually. How much will be in the account immediately after the last payment?

Note that the compounded formula is

$$A = P(1+i)^n$$

Time (years) 0 1 2 3 4 5

Time	1st deposit	2nd deposit	3rd deposit	4th deposit	5th deposit	6th deposit	7th deposit	8th deposit	9th deposit	10th deposit
0	400									
	$400(1.04)$	400								
1	$400(1.04)^2$	$400(1.04)$	400							
	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400						
2	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400					
	$400(1.04)^5$	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400				
3	$400(1.04)^6$	$400(1.04)^5$	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400			
	$400(1.04)^7$	$400(1.04)^6$	$400(1.04)^5$	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400		
4	$400(1.04)^8$	$400(1.04)^7$	$400(1.04)^6$	$400(1.04)^5$	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400	
	$400(1.04)^9$	$400(1.04)^8$	$400(1.04)^7$	$400(1.04)^6$	$400(1.04)^5$	$400(1.04)^4$	$400(1.04)^3$	$400(1.04)^2$	$400(1.04)$	400
5										

Amount of Annuity

$$= 400(1.04)^9 + 400(1.04)^8 + 400(1.04)^7 + 400(1.04)^6 + 400(1.04)^5 + 400(1.04)^4 + 400(1.04)^3 + 400(1.04)^2 + 400(1.04) + 400$$

$$= 400 + 400(1.04) + 400(1.04)^2 + 400(1.04)^3 + 400(1.04)^4 + 400(1.04)^5 + 400(1.04)^6 + 400(1.04)^7 + 400(1.04)^8 + 400(1.04)^9$$

Amount of Annuity

The amount of an annuity =

$400 + 400(1.04) + 400(1.04)^2 + 400(1.04)^3 + 400(1.04)^4 + 400(1.04)^5 + 400(1.04)^6 + 400(1.04)^7 + 400(1.04)^8 + 400(1.04)^9$

The amount of annuity is a geometric series with $a = 400$, $r = 1.04$, and $n = 10$

$$\text{The amount of annuity} = 400\frac{1-(1.04)^{10}}{1-1.04} \approx 4802.44$$

Amount of Annuity

The amount of an annuity =

$R + R(1+i) + R(1+i)^2 + R(1+i)^3 + R(1+i)^4 + R(1+i)^5 + R(1+i)^6 + R(1+i)^7 + R(1+i)^8 + R(1+i)^9$

The amount of annuity is a geometric series with $a = R$ and $r = 1 + i$.

$$A = R\frac{1-(1+i)^n}{1-(1+i)}$$

$$= R\frac{1-(1+i)^n}{-i} = R\frac{(1+i)^n-1}{i}$$

R: monthly payment
i: interest per period
n: number of payments
A: Amount of Annuity

Problem 4 page 853

Find the amount of an annuity that consist of 20 semiannual payments of $500 each into an account that pay 6% interest per year, compounded semiannually.

Present Value of an Annuity

The *present value* of an annuity is the amount P that must be invested now at the interest rate i per time period in order to provide n payments, each of amount R.

Example

A life insurance company offers an optional settlement of cash or \$500 a year for ten years, paid in equal installment at the end of each year. What would be the equivalent cash settlement by the company if money earns at an annual rate of 10% compounded annually?

Time (years) 0 1 2 3 4 5 6 7 8 9 10

	1st \$500 payment	2nd \$500 payment	3rd \$500 payment	4th \$500 payment	5th \$500 payment	6th \$500 payment	7th \$500 payment	8th \$500 payment	9th \$500 payment	10th \$500 payment
0	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$	$500(1.1)^{-1}$
1		$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$	$500(1.1)^{-2}$
2			$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$	$500(1.1)^{-3}$
3				$500(1.1)^{-4}$	$500(1.1)^{-4}$	$500(1.1)^{-4}$	$500(1.1)^{-4}$	$500(1.1)^{-4}$	$500(1.1)^{-4}$	$500(1.1)^{-4}$
4					$500(1.1)^{-5}$	$500(1.1)^{-5}$	$500(1.1)^{-5}$	$500(1.1)^{-5}$	$500(1.1)^{-5}$	$500(1.1)^{-5}$
5						$500(1.1)^{-6}$	$500(1.1)^{-6}$	$500(1.1)^{-6}$	$500(1.1)^{-6}$	$500(1.1)^{-6}$
6							$500(1.1)^{-7}$	$500(1.1)^{-7}$	$500(1.1)^{-7}$	$500(1.1)^{-7}$
7								$500(1.1)^{-8}$	$500(1.1)^{-8}$	$500(1.1)^{-8}$
8									$500(1.1)^{-9}$	$500(1.1)^{-9}$
9										$500(1.1)^{-10}$
10										

Amount of Annuity

$$= 500(1.1)^{-1} + 500(1.1)^{-2} + 500(1.1)^{-3} + 500(1.\ )^{4} + 500(1.1)^{-5} + 500(1.1)^{-6} + 500(1.1)^{-7} + 500(1.1)^{-8} + 500(1.1)^{-9} + 500(1.1)^{-10}$$

$$= 500(1.1)^{-10}[1 + (1.1)^1 + (1.1)^2 + \cdots + (1.1)^{10}] = 500(1.1)^{-10}\,\frac{1-(1.1)^{10}}{-0.1} = 500\,\frac{1-(1.1)^{-10}}{0.1} = \$3072.28$$

Present Value of an Annuity

The present value of an annuity with *n* regular equal payments of size *R* and interest rate *i per time period is given by*

$$P = R(1+i)^{-1} + R(1+i)^{-2} + R(1+i)^{-3} + R(1+i)^{-4} + R(1+i)^{-5} + R(1+i)^{-6} + R(1+i)^{-7} + R(1+i)^{-8} + R(1+i)^{-9} + R(1+i)^{-10}$$

$$P = R\frac{1-(1+i)^{-n}}{i}$$

R: monthly payment
i: interest per period
n: number of payments

Problem on page 853

6. How much money should be invested every quarter at 10% per year, compounded quarterly, in order to have \$5,000 in 2 years?
8. What is the present value of an annuity that consists of 20 semiannual payments of \$1000 at the interest rate of 9% per year, compounded annually.

Installment Buying

Example 4 A student wishes to buy a car. He can afford to pay \$200 per month but has no money for a down payment. If he can make these payments for four years and the interest rate is 12%, what purchase price can he afford?

$$P = R\frac{1-(1+i)^{-n}}{i} = 200\frac{1-(1+.1)^{-48}}{.1} \approx 7594.79$$

Installment Buying

If a loan P is to be repaid in n regular equal payments with interest rate i per time period, then the size R of each payment is given by

$$R = p\frac{i}{1-(1+i)^{-n}}$$

Problem 12 page 853

What is the monthly payment on a 30-year mortgage of \$80,000 at 9% interest? What is monthly payment on this same mortgage if it is to be repaid over a 15-year period?

Practice Problems on Page 853

1 – 17 odds